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Preface

by Professor Dr. Adelina Georgescu, ROMAI President

This volume contains part of the papers presented at the Conference on Applied and Industrial Mathematics (CAIM) held in Oradea in the period May 29-31, 2003. CAIM 2003 represents the eleventh edition of the annual scientific meetings coorganized by Romanian Society of Applied and Industrial Mathematics (ROMAI) and some university. Starting with 1993, the CAIMs were held in Oradea (1993, 1994), Oradea and Chisinau (1995, 1996, 1997) and they were coorganized by ROMAI and Univ. of Oradea (UO), the Technical University of Tiraspol (TUT), Technical University of Moldova (TUM) and Institute of Mathematics and Computer Science of Academy of Sciences of Moldova (IMCS-ASM). In 1996, the Chisinau Session of CAIM was held in the framework of Symposium Septium Tiraspolense. The organization was possible thanks to the generous involvement of the Professor Dr. Teodor Maghiar (Rector), the late Professor Dr. Gheorghe Nadiu and his young group of the Dept. of Mathematics (UO): Ioan Dzitac, Valerian Tibu, Ioan Fechete, Aurel Caus, Emil Schwab, Emilia Borsa, Marian Degeratu; the late Professor Petru P.Osmatescu (TUM), Academician Mitrofan Ciobanu (TUT, actually Rector), Academician Mefodie Rata (IMCS-ASM), Professor Dr. Constantin Gaindric (Director, IMCS-ASM), Professor Dr. Mihail Popa (Vicedirector, IMCS-ASM), Professor Dr. Dumitru Botnaru (TUM) and their young collaborators, Professors Laurentiu Calmutchi and Liubomir Chiriac (TUT).

In the early years '90, to publish the proceedings was a courageous action. Indeed, by that time, the sheets of papers and copiators were rare and their price was high, beyond the possibilities of ROMAI. However, UO or members of ROMAI themselves (e.g. Professors Carmen Rocsoreanu and Nicolae Giurgiteanu) supported the costs for the proceedings, ensuring their in time publication. In spite of their poor graphical quality, these proceedings attested that the scientific life in Romania goes on without interruptions and its level is very high, even if it was achieved with important sacrifices. The papers published in CAIM Proceedings were used, among others, for validation of PhD degrees or for promoting scientific and didactical degrees for their authors.

In the course of time it was more and more evident that the main directions of interest for CAIM participants were firstly: analytical and numerical methods in mechanics, aviation, naval hydrodynamics, meteorology, hydrology, physics; algebra, logic and topology and subsequently: finite- dimensional dynamical systems (y compris fractals and deterministic chaos), applied computer science, education. These directions were the fields of expertise of participants in various branches of mathematics, physics, engineering, biology, economics, computer science (mainly universitary professors and research workers) from: the Institute of Applied Mathematics (IMA, Bucharest), Institute of Mathematics (Chisinau, Bucharest), National Institute for R&D in Material Physics (Bucharest), National Institute for Optoelectronics (Bucharest), Institute of Microtechnology (Bucharest), Institute for Solid Mechanics (Bucharest), Institute of Aviation (Bucharest), Institute for Computer Science (Iasi), Research and Design Institute for Shipbuilding (ICEPRONAV, Galati), Institute for Nuclear Research (Pitesti-Mioveni), Institute of Physics and Nuclear Engineering (Bucharest), Academy of Economical Studies (Chisinau, Bucuresti), Centre of Mathematical Statistics (Bucharest),Institute of Fundamental Technological Research (Warsaw); universities of: Timisoara, Iasi, Bucharest, Craiova, Cluj-Napoca, Oradea, Pitesti, Brasov, Alba Iulia, Arad, Chisinau, Tiraspol, Spiru Haret Univ. (Craiova), Oxford (England), Ibaraki Univ. (Japan), Nihon Univ. (Japan), Le Havre (France), Bari (Italy), Belgrade (Serbia and Montenegro), Galati, Petrosani, Baia Mare, Targoviste, Targu Mures, Cosice (Slovakia), Siedlce (Poland), Kaiserslautern (Germany), Moscow, Ulianovsk (Russia), Tashkent (Uzbekistan), Tokyo (Japan), Montreal (Canada), Paris XII (France), Constanta, Balti, Targu Jiu, Drobeta Turnu Severin, Olomouc (Czech Republic), Athens, Patras (Greece), Leeds (England), Porto (Portugal) and other institutions from: Chisinau, Pitesti, Mioveni-Arges, Curtea de Arges, Koln (Germany), Bucharest, Craiova, Ramnicu Valcea, Sinaia, Cluj-Napoca, Sibiu.

Among them, the strong group of ROMAI members from IMA, Chisinau, Tiraspol, Craiova and, later, Pitesti, configurated the specificity of ROMAI and its CAIMs versus other groups of applied mathematicians and conferences really active in Romania: the group of numerical analysis and optimal

control at Constanta, and the groups of Baia Mare and Timisoara. Also at Chisinau, in 2001, ROMAI members from Republic of Moldova leaded by Academician Mitrofan Ciobanu organized a very good conference where the applied and industrial mathematics was well represented. From the Organizing Committee we quote Parascovia Sirbu and Florin Damian. A special mention is deserved to the fact that the ROMAI branch from Republic of Moldova organized at Chisinau a duplex session with UO in 2001.

The next five CAIMs were held in Pitesti (1998, 1999, 2000, 2001) and in Pitesti and Mioveni-Arges (2002). They have been coorganized by ROMAI and Univ. of Pitesti (UP) and the Local Council and the Cityhall of Mioveni-Arges. In their organization the precious support of Professor Dr. Gheorghe Barbu (Rector), Professor Dr. Marioara Abrudeanu (Vicerector), Eng. Vasile Costescu (Mayor, Mioveni-Arges) and Ion Georgescu (Vicemayor) is kindly acknowledged. At UP, among the organizers we quote: Anca-Veronica Ion, Mircea Bolosteanu, Gheorghe Nistor, Bogdan Nicolescu, Constantin Georgescu, Catalin Ducu, Sebastian Parlac, Nicolae Popa, Florica Raduna, Raluca Heroiu (Georgescu), Cristina-Simona Ion, Aniela Dragomir, Antonio-Mihail Nuica, Elena Codeci, Daniela Cristina Trancau (Sarbu), Marius Macarie. In Mioveni-Arges, the help offered by our former student Liliana Sandulescu was essential. We also mention the unexpected significant support on the part of the group of teachers of mathematics and physics leaded by Camelia-Elena Pufu (Director) and Mariana Radulescu.

The proceedings of these five CAIMs were published in Buletin Stiintific of UP, Series Mathem.-Inf. and their high quality is mainly due to our colleague Professor Lecturer Mircea Bolosteanu. The communications of the section of education from CAIM 2002 were published by Tiparg edition house in a separate volume. In 2002, at Mioveni-Arges, it was for the first time that such a section was successful at a CAIM.

The eleventh edition of CAIM was again coorganized by ROMAI and UO and it was influenced by two great events. The first is the anniversary of the founder of UO, Rector Teodor Maghiar, who always offered his elegant moral and material support to CAIMs. In the name of ROMAI members we address to him all our gratitude and the best wishes for the future.

The second was that the Doctor Honoris Causa title of UO was conferred to two distinguished Romanian mathematicians, Academician Radu Miron and Professor Dr. Gheorghe Micula. In addition, CAIM 2003 benefited to the best e-mail organization in the CAIM history. I have the approval of all participants to CAIM 2003 to express heartily thanks to Associate Professor Dr. Ioan Dzitac for the excellent job he did. He was faced with significant difficulties: thousands of messages, at least three variants of more than 80 written versions of the communications, incertitude of financial supports and many other shortcomings. The majority of these drawbacks could be avoided if the material possibilities of the participants were higher. Professor Dzitac understood this situation and sacrificed his time and forces for the success of CAIM. This is why our thanks to him are so vivid. In his effort, Professor Dzitac was constantly helped and counseled by the Rector Maghiar. Also, he enjoyed the collaboration of his young colleagues, Horea Oros and Daniel Erzse. Besides the former group of organizers from the UO, Professor Dr. Mircea Balaj, the head of the Dept. of Mathematics of UO, together with other members of this department contributed also to the very good organization of CAIM 2003.

Under these auspices, the CAIM 2003 was a real success: very high quality of communications including some of the most recent topics in applied mathematics, industrial mathematics, pure mathematics and computer science.

Like to all previous editions, the section of algebra, logic and topology was very well represented. Differential geometry was the concern of some very good communications. A lot of papers dealt with equations (integro-differential, integral, functional, stochastic) and their qualitative analysis (e.g. stability) and numerical methods for them. Complex applications to hydrodynamics, hydromagnetic stability, flows through porous media, electrodynamics, mechanics, engineering (aviation, electronics, shipbuilding), economics, biology, medicine were presented. Interesting papers on asymptotic analysis, complex analysis, functional analysis, bifurcation were exposed. Dynamical systems (and, in particular, neural networks) were treated too. We quote some communications on fractional programming, operational research and mathematical statistics. Definitely, the novelty at this CAIM was represented mainly by computer science (parallel computing, security in e-mail system Internet-based), image processing,

numerical modeling in ecology (e.g. dynamic process above Central Asia). In addition, we quote two highly interesting papers on application of asymptotic analysis in complex field of the Schrodinger equation. Finally, we remark the increasing interest for the modern mathematical training of the new generations expressed in the communications presented at the education section of CAIM 2003.

Looking at the number and expertise of CAIM participants, mainly ROMAI members, it follows that ROMAI became the most prestigious society of Romanian applied and industrial mathematics and that CAIMs are already famous not only in Romania. Some of them are the authors of the articles herein. Renowned universitary professors or research workers from abroad participated to CAIMs: Allen Tayler (the founder and President of European Consortium of Mathematics in Industry), Catherine Bandle (ROMAI member, for a long time Director of the Department of Mathematics of Univ. of Basel, Switzerland), Zbigniew Peradzynsky and Kazimierz Piechor (Institute of Fundamental Technological Research, Warsaw), Vladilen Trenogin (Academician, former Director of Moscow Steel and Alloys Institute), Mitrofan Ciobanu (Academician, President of Society of Mathematics from Rep. of Moldova), Mefodie Rata (Academician, President of the Committee for Promoting Mathematicians of Rep. of Moldova), Boris Loginov (Univ. of Ulyanovsk, Russia), Mirsaid Aripov (National University of Uzbekistan,Tashkent), Nenad Mladenovic (President of Society of Applied and Industrial Mathematics, Serbia and Montenegro), Kiyoyucki Tchizawa (Musashi Tech. Univ., Japan), Lidia Palese (Univ. of Bari), Maria Margarida Amorin Ferreira (Univ. of Porto), Maria do Rosario de Pinho (Univ. of Porto), Anthippi Poulkou (Univ. of Athens), Chrysoula Kokologiannaki (Univ. of Patras), Harry Vereecken (Director of Institute of Chemistry and Dynamics of the Geosphere, Julich).

This volume contains part of the communications delivered to CAIM 2003. The choice was dictated by their Latex 2e presentation and the preference for the English language. A second volume will contain the rest of the communications written in Word and mostly in Romanian language.

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Numerical modeling of dynamic processes above Central Asia

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1. Introduction

On the basis of 15 leveled regional of non-adiabatic model [1] based on the prediction of a complete system of equations of hydro thermodynamics, the modeling of winter dynamic processes over Central Asia is carried out. Moreover, the model intended for modeling large-scale processes is used for the diagnosis of mezoscale phenomena such as fronts. Since these types of models do not reproduce with fidelity the structure of front, the frontogenetic function was chosen as the parameter, on which it is possible to decide about the presence of developing front.

Distinctive features of the Central Asian region (CAr) are the high mountain files bending around the CAr from the south and east and the drying up Aral Sea, resulting in ground drying up, and formation of deserts and dusty storms. Being a barrier against cold winds from north and hot winds from the south, the Aral sea plays the important role in mitigation of climatic system of region and serves as the catalyst for the formation of clouds because of huge weights of water pair, rising from its surface. At the end, this moisture renews stocks of ice and snow on the distant mountain slopes, thus finishing the water circulation.

The modeling was carried out on the GARP data. The linear change of borders with 12-hour interval is stipulated in the model.

Previously the processes dominant in wintertime on investigated region were analyzed and selected.

The qualitative comparative analysis of actual and prognostic cards, which have been carried out with the account of macro-synoptic condition, shows a good agreement of actual and model data if complex processes in atmosphere - formation and evolution of cyclone and aggravation of front and anticyclogenesis occur.

The same conclusion is confirmed by quantitative estimations of modeling successiveness.

On the account of frontogenetic function it is necessary to calculate the derivatives from nonlinear members, vertical speeds, sources and drains. This imposes high requirements to initial information. In order to suppress the possible noise in data it is convenient to use median filtration, which is one of methods of nonlinear signal processing. The computations showed that the median filter substantially suppresses noises, allocates useful signal, leaving constant its place.

The comparison of actual and forecasting cards of frontogenetic function shows that the model is able informatively to describe its basic changes within one day.

Only on the basis of prognosis results comparison of complex processes on the region together with synoptic analysis is possible to decide on the practical applicable features of frontogenetic functions and on the method of diagnosis of fronts.

2. Problem formulation and method of solution

On the basis of classification of atmospheric processes above CAr, the following dynamic processes dominant in winter time above the investigated region (0° w.l.- 90° e.l., 67,5° n.l.-20° n.l.) the development of which and an aggravation of frontal zones is typical were found: 1) January, 2-4 ? a development of the western cyclone; 2) January, 4-6 - the western cyclone with wave activity on fronts; 3) January, 5-7 - the western carry, occurrence of a wave; 4) January, 7-9 - strengthening of a wave, development of the Murgab cyclone; 5) January, 10-12 - southwest periphery of an anticyclone; 6) January, 21-23 - southwest periphery of an anticyclone, development of the south Kaspii cyclone with the subsequent western and northwest intrusion into Central Asia; 7) January, 23-25 - northwest streams, southern periphery of an anticyclone, development of the Murgab cyclone.

A distinctive feature of the CAr is the complex lay of land. So, from a southwest Central Asia the Caucasian mountains, from the south - bend around Kopet Dagh and spurs of Hindu Kush with heights more than 3 km, from the east - Tien Shan and Pamir with heights 5-7 km.

Modeling was carried out on the basis of 15 level regional, diabatic model of the atmosphere [1] based on the dprediction of a full system of the hydro thermodynamics equations. Numerical integration of the model and parameterization of physical mesoscale processes, are described in detail in [1].

The linear change of borders with a 12-hour interval in model is used. Term of integration ?: 48h.

2.1. **Numerical results.** For the general representation of quality of reproduction the model of the specified processes averaging on all situations factors of correlation (K), relative error (E) (tab. 1) and full Lorenz energetic cycle were calculated. According with table 1, the quality of modeling is high.

Table 1 Estimations of quality of modeling of winter dynamic processes above CAr

Before to proceed to the qualitative analysis of the results of modeling, it is necessary to choose the parameter caracterizing the presence of developing front. By studying the change in due course of this parameter, it is possible to decide on the character of the physical processes which take place in an atmosphere. Such a characteristic is the frontogenetic function (F), determined as a time derivative of a horizontal gradient of temperature in a moving particle.

To positive values will correspond process of strengthening of horizontal gradients of temperature, e. c. frontogenesis, negative - frontolis.

Since for the computations of the frontogenetic function it is necessary to compute the derivatives from nonlinear members, vertical speeds, sources and drains, high requirements to the initial data must be imposed. Indeed, a series of the computations which have been carried out on GARP dataset, showing high sensitivity frontogenetic function to possible noise in given measurements (fig. 1 a, b). In addition the noise level quite often proves to be equal to a level of a useful signal. In order to suppress the noise the filtration, which is one of methods of processing nonlinear signals the median filter is used by us. The median filtration keeps sharp differences in fields whereas the usual linear filter smooth these differences.

A series of experiments with data GARP by means of median filter has shown that the best results are obtained when using the filter with the aperture 5X5 and when the field of temperature is exposed to a filtration only. The median filter substantially suppresses noise, allocates a useful signal, leaving constant its location.

Now let us proceed to the qualitative analysis of dynamic processes. Below, as an example, the experiment 2 is described.

In order to study the cyclones and fronts evolution, fields H-1000, H-500, T-1000 , H-850 were designed and analyzed. The choice of these fields is dictated by the fact that they are the traditional basis in working with weather forecasters.

Thus, during numerical experiment with 00 h January, 4 up to 12 h January, 5 (fig. 1) the following changes were essential:

1) the high-altitude cyclone above the Baltic coast of Poland has ceased to be inactive and the speed of 15-20 km/h is began displaced to a southeast to Warsaw - Kiev - Moscow;
2) as a result of the rapprochement of three air weights there was an increase of gradients along all site Planetary high-altitude frontal zone to the east from 40° e.l.;
3) along strips 45-55° n.l. from Warsaw up to Southern Ural activation of cyclonic activity with a deepening available and formation of the new cyclones centers at the ground was observed. East center of depression deepens due to advection of heat up to 12h January, 4, sometimes remained without changes. As a result occlusion began filled. The most western cyclone formed at the ground, all period of numerical experiment, slowly going deep, approached of a cyclone. In 00h January, 5 both centers were linked by one closed isobar of 1000mb. However, then, due to a non-uniform growth of pressure, they have again splitted in two centers;
4) in the rear of the system consisting of two cold longitude focused fronts above Asia Minor peninsula, an anticyclone began formed. The growth in pressure above Caucasus has reached

5mb/3h, above Turkmenistan 7-9mb/3h. Such an active anticyclogenesis has caused the big changes in weather conditions in CAr: a strong wind, deposits.

This brief synoptic analysis yields an illustration of the macroscale situation on the basis of it is more convenient to study mesoscale features of the situation chosen for the experiment.

Let us now discuss the results of the numerical experiment, following the above synoptic analysis. The high-altitude moving cyclone, former up to 12h January, 3 inactive in area of Warsaw, all over again to a southeast, then to the east and to northeast, is well predicted by the model. Connected with a deepening of a cyclone and its displacement to the south increase of gradients at the southern periphery of this cyclone, the amplification of a crest to the east from a cyclone in 12 h January, 4 and in 00 h January, 5 is also reflected by model. It is only at 12 h January, 5 that the model, having correctly reflected the tendency of destruction of east crest, "was mistaken" in the speed of its destruction. The model reflects increasing gradients along subtropical Planetary high-altitude frontal zone to the east from 40 ° e.l. Development of ground depression (on the real and modeling data) is shown in on fig. 2 (d, e) and (b, g). On these figure it is seen that the changes in the form, structures and intensity of multicenter formations were complex. The basic changes are: its deepening owing to advection with the subsequent filling occlusion east center and a deepening of the western center model is reflected. Anticyclogenesis above Minor Asia, branch of the closed anti-cyclonic nucleus from a subtropical crest and fast moving of the generated anticyclone through Caucasus and CAr are also well described by the model.

In a southeast of region the model reproduces a crest which is not observed on real maps. The reason of such behavior of model is the complex orography of this region.

The qualitative comparative analysis of the actual and modeling cards, carried out on the account macrosynoptic conditions, has shown a good agreement of the actual and modeling data if in the atmosphere there were very complex processes: formation and evolution of a cyclone, an aggravation of front and anticyclogenesis.

On fig. 1 fields of frontogenetic are shown to act on a surface 1000 GPa and position of frontal systems. Even simple comparison of values of function F on two surfaces (H1000 and H500) reveals the ambiguity and complexity of its space distribution.

The analysis of materials of the given experiment shows that the fields of the frontogenetic function contain the information necessary to the explanation of evolution of fronts and related formations. Thus, on the maps for 00 January, 4 it is seen, that big positive values F both at the ground and in average troposphere are near to peaks of the developing waves of the cold front, which further has resulted in the development of these waves and registration of a cyclone. Concurrence of areas raised values F in area 60° n.l. with 70-80°e.l. On surfaces 1000 and 500 GPa has also resulted a further formation of a cyclone from a flat wave.

The area of negative values frontogenetic functions on both surfaces lies in the range of 10-20° e.l., 30-50° n.l., accompanied by the appropriate wave indignation on the cold front owing to the fact that what this wave has not received a further development.

Lowering Arctic Planetary high-altitude frontal zone in 12 h January, 4 has resulted in already marked aggravation of fronts along 50° n.l. and to the east 40° e.l. Is especially obvious on a surface 1000 GPa. The greatest positive values F are marked along a zone of the Arctic front in 12 h January, 4 and in 00 h January, 5 and the peaks of an internal wave it begun the occlusion of a cyclone.

Position of areas of positive values of F and change of their intensity is rather easy and obviously possible to connect with certain sites of the fronts. Areas of negative values of F coincide with areas of divergence on a surface 500GPa, or with the position antycyclonic crosspieces - saddles at the ground. The concurrence of maximal positive values of F to position of the frontal sections and peaks of the waves on them 12 h after is important for the forecasting. In the given experiment this feature is distributed over all four terms of integration. Comparison actual and modeling maps of the frontogenetic function shows that the current model of the first day is able to describe its basic changes.

3. Conclusion

On the basis of the results of numerical modeling of dynamic processes above Central Asia region it is possible to draw the following conclusions:

1 the model reproduces well the evolution fields of temperature, wind and vertical speed during 1.5 days;
2 the frontogenetic function describes effectively the development of the frontal areas during 24h;
3 the comparison of the separate components of the frontogenetic functions reveals a bend isentropic surfaces in the bottom troposphere at certain times more than its other components. The contribution of the heat inflowsto the total frontogenetic function proved to be less than its other components.

References

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Remarks on varieties of compact rings

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Let $\mathfrak{K}$ be a class of Hausdorff compact rings. $\mathfrak{K}$ is called *variety* if it is multiplicative, hereditarily closed and closed under taking of continuous homomorphic images.

Let $\mathfrak{M}$ be a variety of compact rings and $\mathfrak{K}$ a subclass of it. We say that $\mathfrak{M}$ is generated by $\mathfrak{K}$ if $\mathfrak{M}' \supseteq \mathfrak{K}$, $\mathfrak{M}'$ is a subvariety of $\mathfrak{M}$ implies $\mathfrak{M}' = \mathfrak{M}$.

Theorem 1. *If $\mathfrak{K}$ is a class of compact rings, then $PQSP(\mathfrak{K}) = QSP(\mathfrak{K})$, $SQSP(\mathfrak{K}) = QSP(\mathfrak{K})$, $QQSP(\mathfrak{K}) = QSP(\mathfrak{K})$.*

Proof. We will show that $PQSP(\mathfrak{K}) = QSP(\mathfrak{K})$. Indeed, let $A_i \in QSP(\mathfrak{K})$, $i \in I$. There exist $B_i \in SP(\mathfrak{K})$ and continuous surjective homomorphisms $\varphi_i : B_i \to A_i$. Each B_i is isomorphic to a closed subring of a ring P_i, which is a topological product of rings from $\mathfrak{K}$. We identify B_i with its image in P_i. Since $\prod B_i \subseteq \prod P_i$ is a closed subring of $\prod P_i$, we obtain that $\prod B_i \in SP(\mathfrak{K})$. Since $\prod A_i$ is a continuous homomorphic image of $\prod B_i$, we obtain that $\prod A_i \in QSP(\mathfrak{K})$. We proved that $PQSP(\mathfrak{K}) = QSP(\mathfrak{K})$.

We will show that $SQSP(\mathfrak{K}) = QSP(\mathfrak{K})$. Let $A \in SQSP(\mathfrak{K})$, i.e., A is topologically isomorphic to a closed subring of a ring $B \in QSP(\mathfrak{K})$ (again we identify A with its image). There exists $H \in SP(\mathfrak{K})$ and a continuous surjective homomorphism $\varphi : H \to B$. Evidently, A is a continuous homomorphic image of the closed subring $\varphi^{-1}(A)$ of H and $\varphi^{-1}(A) \in SP(\mathfrak{K})$. It follows that $A \in QSP(\mathfrak{K})$ and so $SQSP(\mathfrak{K}) = QSP(\mathfrak{K})$.

The relation $Q^2(\mathfrak{K}) = Q(\mathfrak{K})$ implies $QQSP(\mathfrak{K}) = QSP(\mathfrak{K})$. □

Theorem 2. *If $\mathfrak{K}$ is a class of compact rings, then $QSP(\mathfrak{K})$ is the smallest variety of compact rings containing $\mathfrak{K}$.*

Proof. Let $\mathfrak{M}$ be a variety of compact rings containing $\mathfrak{K}$. Then $\mathfrak{M} \supseteq QSP(\mathfrak{K})$ which is a variety by the last theorem. □

We give here some examples of varieties of compact rings.

1. The class of all compact rings.
2. The class of compact commutative rings.
3. If $\mathfrak{M}$ is an abstract variety of rings, then the class of all compact rings which belongs to $\mathfrak{M}$ is a variety of compact rings.
4. The class of all compact p-rings, where p is prime.
5. The class of zero-dimensional compact rings.
6. Let X be a Tychonoff space. A compact ring R is called *the free compact ring generated by X* if are satisfied the following conditions:

i) X is a subspace of R

ii) R is topologically generated by X

iii) for each continuous function $f : X \to R'$ there is a continuous homomorphism $\widehat{f} : R \to R'$ extending f.

It can be proved that for any space X there exists the free compact ring $F(X)$ generated by X.

Let now X be any discrete space and M any subset of $F(X)$. Denote by $V(M)$ the class consisting of compact rings R with property that for any continuous homomorphism $h : F(X) \to R$, $h(M) = 0$. Then $V(M)$ is a variety of compact rings.

Let I be a closed two sided closed ideal of $F(X)$. Denote by $\mathfrak{M}(I)$ the class of all compact rings R with property that $I \subseteq \ker(f)$ for every continuous homomorphism $f : F(X) \to R$.

We shall prove that the class $\mathfrak{M}(I)$ is a variety of compact rings. Indeed, let $R_\alpha \in \mathfrak{M}(I)$, $\alpha \in \Omega$. Let $f : F(X) \to \prod R_\alpha$ be a continuous homomorphism. Then $\mathrm{pr}_\alpha \circ f(I) = 0$ for each $\alpha \in \Omega$. It follows that $f(I) = 0$, i.e. $\prod R_\alpha \in \mathfrak{M}(I)$. Let S is a closed subring of a ring $R \in \mathfrak{M}(I)$ and

$f : F(X) \to S$ a continuous ring homomorphism. Then f can be considered (in an obvious way) as a homomorphism of $F(X)$ in R. It follows that $f(I) = 0$, hence $S \in \mathfrak{M}(I)$. Let $\alpha : R \to R'$ a continuous homomorphism of a compact ring $R \in \mathfrak{M}(I)$ on a ring R'. Since α is surjective, for each $x \in X$ the set $\alpha^{-1}(f(x))$ is non-void; choose $t_x \in \alpha^{-1}(f(x))$. The mapping $X \to R$, $x \mapsto t_x$ can be extended to a continuous ring homomorphism $\widehat{f} : F(X) \to R$. We claim that $f = \alpha \circ \widehat{f}$. Indeed, if $x \in X$ then $\alpha \circ \widehat{f}(x) = \alpha(t_x) = f(x)$. Since X is a topological generating set for $F(X)$, we obtain that $\alpha \circ \widehat{f} = f$. Then $f(I) = \alpha \circ \widehat{f}(I) = 0$, therefore $R' \in \mathfrak{M}(I)$.

7. Denoted by $\mathfrak{M} = Var(\mathbb{Z}_p)$ the variety of compact rings generated by $\mathbb{Z}_p$ ($\mathbb{Z}_p$ is the ring of p-adic integers). Then any ring from $\mathfrak{M}$ is a commutative compact p-ring. There are finite commutative p-rings which are not in $\mathfrak{M}$. For example, $GF(p^2) \notin Var(\mathbb{Z}_p)$. Indeed, for any $x \in \mathbb{Z}_p$, $\lim_{n\to\infty}(x^p - x)^n = 0$, therefore this condition is true for any ring from $Var(\mathbb{Z}_p)$. Since $GF(p^2)$ doesn't satisfy the identity $x^p - x = 0$ it is not contained in $Var(\mathbb{Z}_p)$.

Question. Is true that any finite commutative nilpotent p-ring is in $Var(\mathbb{Z}_p)$?

8. The class of compact rings which admit the Wedderburn-Mal'cev decomposition is a variety.

Definition 3. *We will say that the compact ring R admits the Wedderburn-Mal'cev decomposition if there exists a closed subring S such that $R = S \oplus J(R)$.*

Lemma 4. *If $\{R_\alpha : \alpha \in \Omega\}$ is a family of compact rings which admit the Wedderburn-Mal'cev decomposition then the product $\prod R_\alpha$ admits the Wedderburn-Mal'cev decomposition too.*

Proof. Indeed, $\prod R_\alpha = \prod S_\alpha \oplus J(\prod R_\alpha)$ and $J(\prod R_\alpha) = \prod J(R_\alpha)$. □

Lemma 5. *If R is a compact ring which admits the Wedderburn-Mal'cev decomposition and R' is a closed subring, then R' admits too the Wedderburn-Mal'cev decomposition.*

Lemma 6. *If R is a compact ring admitting the Wedderburn-Mal'cev decomposition and R' is its continuous homomorphic image, then R' admits the Wedderburn-Mal'cev decomposition too.*

Proof. Let $f : R \to R'$ and $R = S \oplus J(R)$; then $R' = f(S) + f(J(R))$ and $f(J(R)) \subseteq J(R')$, therefore $R' = f(S) + J(R')$. Since $f(S)$ is regular and compact $R' = f(S) \oplus J(R')$. □

We will extend the notion of a product of two varieties to the topological case [N], [7].

Definition 7. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two varieties of compact rings. Then the product $\mathfrak{M} \circ \mathfrak{N}$ of varieties $\mathfrak{M}$ and $\mathfrak{N}$ consists of those compact rings R for which there exists a closed ideal $I \in \mathfrak{N}$ such that $R/I \in \mathfrak{M}$.*

We will show that $\mathfrak{M} \circ \mathfrak{N}$ is indeed a variety. Let $R_i \in \mathfrak{M} \circ \mathfrak{N}$, $i \in I$; then for any $i \in I$ there exists $I_i \in \mathfrak{N}$ such that $R_i/I_i \in \mathfrak{M}$. Since $\prod I_i \in \mathfrak{N}$ and $\prod (R_i/I_i) \cong_{top} \prod R_i / \prod I_i \in \mathfrak{M}$, we get that $\prod R_i \in \mathfrak{M} \circ \mathfrak{N}$. We obtained that $\mathfrak{M} \circ \mathfrak{N}$ is multiplicative.

If $R \in \mathfrak{M} \circ \mathfrak{N}$ then there exists $I \in \mathfrak{N}$ such that $R/I \in \mathfrak{M}$. Let S be a closed subring of R. Then $S \cap I \in \mathfrak{N}$ and $S/S \cap I \cong_{top} S + I/I \in \mathfrak{M}$, hence $S \in \mathfrak{M} \circ \mathfrak{N}$.

Let $f : R \to R'$, $R \in \mathfrak{M} \circ \mathfrak{N}$, then there exists $I \in \mathfrak{N}$ such that $R/I \in \mathfrak{M}$. Then $f(I) \in \mathfrak{N}$ and $R'/f(I) \cong_{top} R/I + \ker f$. Since $R/I + \ker f$ is a continuous image of R/I, we obtain that $R/I + \ker f \in \mathfrak{M}$. We get that $R' \in \mathfrak{M} \circ \mathfrak{N}$.

Consider now the variety $\mathfrak{N}$ of topologically nilpotent rings. It has the property $\mathfrak{N} \circ \mathfrak{N} = \mathfrak{N}$. If $\mathfrak{M}$ is the variety of compact rings admitting the Wedderburn-Mal'cev decomposition, then $\mathfrak{M} \circ \mathfrak{M} \neq \mathfrak{M}$ [consider the ring $\mathbb{Z}/4\mathbb{Z}$].

Theorem 8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ two varieties of compact rings and $\mathfrak{M}_n$, $n \in \mathbb{N}^+$ the variety of compact rings satisfying the identity $x^n = x$. Then $(\mathfrak{A} \circ \mathfrak{M}_n) \circ \mathfrak{B} = \mathfrak{A} \circ (\mathfrak{M}_n \circ \mathfrak{B})$.*

Proof. "$\supseteq$" Obvious.

"$\subseteq$". Let $R \in (\mathfrak{A} \circ \mathfrak{M}_n) \circ \mathfrak{B}$, then there exists an ideal I of R, $I \in \mathfrak{A} \circ \mathfrak{M}_n$ such that $R/I \in \mathfrak{B}$. There exists an ideal I' of I, $I' \in \mathfrak{A}$ such that $I/I' \in \mathfrak{M}_n$.

The factor ring I/I' is semisimple therefore it has an identity. Let $e' = e + I'$ be the identity of I/I'. We can assume that $e^2 = e$; then $I \subseteq eI + I'$.

We claim that I' is ideal in R. Indeed, if $x \in R$ and $i' \in I'$, then $xi' \in I$. Therefore $xi' = ei + i''$, where $i'' \in I'$. It follows that $exi' = ei + ei''$ hence $ei \in I'$, so $xi' \in I'$. In analogous way $i'x \in I'$. Then I' is an ideal in R, $R/I \in \mathfrak{B}$ and $R/I \cong_{top} R/I'/I/I'$, therefore $R \in \mathfrak{A} \circ (\mathfrak{M}_n \circ \mathfrak{B})$. $\square$

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On the multi-state system reliability computing in a MRP/Galerkin framework

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Abstract. The renewal of multi-state systems is modeled using Markov renewal processes. The Markov renewal equation is solved in a Galerkin framework based on trigonometric second order B-splines.

1. Introduction

The evolution of a large class of systems naturally ends when the first failure occurs, but for the others, called multi-state systems, renewal possibilities must be analyzed [4]. This paper presents a mathematical model and its numerical solution in order to compute the steady-state availability and the reliability when the system is complex enough to identify many states from a full operational state to a full under repair state. For the modeling purposes, the Markov Renewal Process (MRP) concept described in [5] and [6] is used. The numerical solution is obtained by a Galerkin approach based on the second order trigonometric B-splines.

The results extend the modeling aspects described in [11]. The approach is different from [10] and incorporates recent developments in Weibull modeling and numerical computing, provided by [8] and [2].

The MRP model is described in the next section. Trigonometric B-splines and the Galerkin approach are considered in Section 3. Finally, numerical experiments are described and concluding remarks are provided (Section 4).

2. The MRP model

Let $0 = T_0 \leq T_1 \leq T_2 \leq \cdots$ be the time instants of successive state transitions in a stochastic process (having K states denoted by: 1, 2, ..., K), and let the random variables X_0, X_1, X_2, $\cdots$ which denote the states of the system at time T_i, $i = 0, 1, \cdots$. Two special states are important: s (start) - for the state *full operational system*, and e (end)- corresponding to a full no-operational system which is *under repair*. In applications, the state s is indexed by 1, and the state e is indexed by K.

Let us assume that at T_0, T_1, $\cdots$, the probability of any particular future behavior of the process, when its current state is known, is not altered by additional knowledge concerning the past (the process history is not important). These instants are called *Markov renewal moments*, and the bivariate stochastic process $(X, T) = \{X_n, T_n; n = 0, 1, 2, \cdots\}$ is a *Markov renewal sequence (MRS)* since it satisfies

$$\begin{aligned} &P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i, \cdots, X_0; T_n, \cdots, T_0\} = \\ &= P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i\} \end{aligned} \tag{1}$$

for all $n = 0, 1, \cdots; i, j = 1, 2, 3, \cdots, K$ and $t \geq 0$.

The state transitions only happen at Markov renewal moments. The underlying stochastic process of the approximate model is a *semi-Markov process (SMP)*, $\{Y = Y(t), t \geq 0\}$ with embedded MRS (X, T). The states of the SMP are defined by $Y(t) = X_n$, if $T_n \leq t \leq T_{n+1}$.

The stochastic process Y is determined by a vector of initial state probabilities $p(0) = [P\{Y(0) = i\}, i = 1, 2, \cdots, K] = [1, 0, \cdots, 0]$ and the kernel matrix $M(t) = [M_{ij}(t)]_{1 \leq i,j \leq K}$, where $M_{ij}(t) = P\{X_s = j, T_s \leq t | X_0 = i\}$ are the conditional transition probabilities of the embedded MRS.

The only non-null elements of matrix $M(t)$ correspond to the possible state transitions in a single step. In practical applications, $M(t)$ is a sparse matrix.

The transient solution of the SMP proceeds by determining the matrix of conditional transition probabilities $V(t) = [V_{ij}](t)$ with $V_{ij} = P\{Y(t) = j|Y(0) = i\}$ which satisfies the following integral equations

$$V_{ij}(t) = [1 - H_i(t)]\delta_{ij} + \sum_{p=1}^{K} \int_0^t V_{pj}(t-s)dM_{ip}(s), \tag{2}$$

where, if $i = j$ then $\delta_{ij} = 1$, otherwise $\delta_{ij} = 0$. Also H_i is given by: $H_i(t) = P\{T_s \leq t|X_0 = i\} = \sum_{j=1}^{K} M_{ij}(t)$.

The set of above equations, for $1 \leq i, j \leq K$ form the so-called *Markov Renewal Equation* (MRE). The unconditional state probabilities $p(t) = [p_i(t)|i = 1 \dots K]$ can be determined as: $p(t) = p(0)V(t)$. Hence, the instantaneous availability $A(t)$ can be easy obtained: $A(t) = 1 - p_e(t)$.

In order to compute the system reliability, let us define the counting process: $\{N(t), t \geq 0\}$, where $N(t) = \sup\{i \geq 0|T_i \leq t\}$ and $\{N_i(t), t \geq 0\}$, where $N_i(t) = |\{X_p = i|0 \leq p \leq N(t)\}|$.

Let be $F(t) = [F_{ij}(t)]_{1 \leq i,j \leq K}$, where

$$F_{ij}(t) = M_{ij}(t) + \sum_{p=1, p \neq j}^{K} \int_0^t M_{ip}(t-s)dF_{pj}(s). \tag{3}$$

The system reliability [3] can be easily determined once $F_{se}(t)$, the first passage time distribution from state s (as initial state) to state e (the system failure state), is computed: $R(t) = 1 - F_{se}(t)$. Other variables of interest in dependability analysis can be computed. If MTTF is an acronym for *Mean Time To Failure*, and MTTR describes the *Mean Time To Repair* variable, then $MTTF = E\{N(t)\}$ and $A = MTTF/(MTTF + MTTR)$ is the steady-state availability [4]. In the context of this paper, MTTR is an input characteristic of the system under study.

3. On solving the Markov renewal equation in a Galerkin-trigonometric B-spline framework

3.1. **Trigonometric B-splines.** Let $\{q_j\}$ be a non decreasing sequence of real numbers such as: $q_j < q_{j+k} < q_j + 2\pi$ and let $k \geq 2$ be an integer. The real functions $T_{j,k}$, which are *right continuously*, have *local support*, and are generated by the recurrence relation [2]

$$T_{j,k} = \begin{cases} \frac{\sin\left(\frac{x-q_j}{2}\right) T_{j,k-1}(x) + \sin\left(\frac{q_{j+k}-x}{2}\right) T_{j+1,k-1}(x)}{\sin\left(\frac{q_{j+k}-q_j}{2}\right)}, q_{j+k} > q_j, \\ 0, \text{otherwise} \end{cases} \tag{4}$$

with

$$T_{j,1}(x) = \begin{cases} \frac{1}{\sin\left(\frac{q_{j+1}-q_j}{2}\right)}, & q_j \leq x < q_{j+1} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

are called *the k-th order trigonometric B-spline functions* based on the sequence knots $(q_i)_{i=j}^{j+k}$. The functions

$$N_{j,m} = \sin\left(\frac{q_{j+m} - q_j}{2}\right) T_{j,m}(x)$$

are called *the normalized trigonometric B-spline functions*.

In this paper only second order trigonometric B-splines are used for numerical experiments. For this reason, let $v \in N^* - \{1\}, k = 2$, the control points $-\pi/v < 0 < \pi/v$ and $w := \pi/(2v)$. Then the

trigonometric B-spline functions of the second order are given by

$$(6)\qquad T_j^{(v)}(x) = \begin{cases} \frac{\sin\left(w\frac{x-q_{j-1}}{q_j-q_{j-1}}\right)}{\sin(w)}, & x \in [q_{j-1}, q_j) \\ \frac{\sin\left(w\frac{q_{j+1}-x}{q_{j+1}-q_j}\right)}{\sin(w)}, & x \in [q_j, q_{j+1}) \\ 0, & \text{otherwise.} \end{cases}$$

3.2. **The Galerkin approach.** Let [0, T] be the time interval under study. In order to apply the Galerkin-trigonometric B-spline method to solve the renewal Markov equation, let us divide the time interval [0, T] into a suitable number, n, of subintervals $[q_i, q_{i+1}], 0 \leq i < n-1$, not necessarily of equal length, with $q_0 = 0$ and $q_n = T$. As previously defined, every second order trigonometric B-spline is considered on the interval $[q_{j-1}, q_{j+1}]$. Let v be a fixed value. Considering $T_l(.)$ as the second order normalized trigonometric B-spline of type v with the knots q_{j-1}, q_j and q_{j+1}, the approximate trigonometric B-spline solution of the above equations can be expressed as linear combinations of the normalized trigonometric B-splines $(0 \leq i, j \leq n-1)$

$$(7)\qquad V_{ij}(t) = \sum_{l=0}^{n} \alpha_l^{ij} T_l(t),$$

and

$$(8)\qquad F_{ij}(t) = \sum_{l=0}^{n} \beta_l^{ij} T_l(t),$$

with α_l^{ij} and β_l^{ij} unknowns to be obtained. The same number of equations can be obtained by using the integral equations and taking the dot product, for $l = 0, 1, .., n$, according to Galerkin method [9]

$$(9)\qquad < T_l, V_{ij} >=< T_l, [1 - H_i(.)]\delta_{ij} > + \sum_{p=1}^{K} < T_l, \int V_{pj}(t-s) dM_{ip}(s) >$$

and

$$(10)\qquad < T_l, F_{ij} >=< T_l, M_{ij} > + \sum_{p=1, p\neq j}^{K} < T_l, \int M_{ip}(t-s) dF_{pj}(s) >,$$

where $0 \leq i, j \leq n-1$ and $< f, g >= \int_0^T f(t)g(t)dt$.

To illustrate the approach, let us consider a system having $K = 3$ states $(s = 1, e = 3)$, with the kernel matrix

$$(11)\qquad M(t) = \begin{pmatrix} 0 & M_{12}(t) & 0 \\ M_{21}(t) & 0 & M_{23}(t) \\ M_{31}(t) & 0 & 0 \end{pmatrix},$$

where M_{ij} are functions describing the state transitions of a particular system under study. According to (2) and (3), the following sets of functions are used

$$V_{11}(t) = 1 - H_1(t) + \int_0^t V_{21}(t) dM12(t-s),\ V_{12}(t) = \int_0^t V_{21}(s) dM_{12}(t-s),$$
$$V_{13}(t) = \int_0^t V_{23}(s) dM_{12}(t-s),\ V_{21}(t) = \int_0^t V_{11}(t) dM_{21}(t-s) + \int_0^t V_{31}(t) dM_{23}(t-s),$$
$$V_{22}(t) = 1 - H_2(t) + \int_0^t V_{12}(t) dM_{21}(t-s),\ V_{23}(t) = \int_0^t V_{13}(t) dM_{21}(t-s) + \int_0^t V_{33}(t) dM_{23}(t-s),$$
$$V_{31}(t) = \int_0^t V_{11}(t) dM_{31}(t-s),\ V_{32}(t) = \int_0^t V_{12}(t) dM_{31}(t-s),\ V_{33}(t) = 1 - H_3(t) + \int_0^t V_{13}(t) dM_{31}(t-s),$$

and

T	$A(T)$	$R(T)$	$F_{13}(T)$	$N(T)$
1,25	0,993456463	1	0	3700
1,66	0,993456463	1	0	3700
2,08	0,999361069	0,998874537	0,001125463	44426
2,50	0,997268763	0,993402041	0,006597959	9094
2,91	0,995917869	0,988336076	0,011663924	6001
3,33	0,995045666	0,983736589	0,016263411	4919
3,75	0,994482529	0,979570036	0,020429964	4405
4,16	0,994118941	0,975767977	0,024232023	4127
4,58	0,993884191	0,972256028	0,027743972	3965
5,00	0,993732625	0,968966699	0,031033301	3867
5,42	0,993634766	0,965843871	0,034156129	3806
5,83	0,993571584	0,962843227	0,037156773	3768
6,25	0,993530791	0,959930944	0,040069056	3744
6,66	0,993504452	0,957081822	0,042918178	3728
7,08	0,993487447	0,954277424	0,045722576	3718
7,50	0,993476468	0,951504447	0,048495553	3712
7,91	0,993469379	0,948753402	0,051246598	3708
8,30	0,993464802	0,946017576	0,053982424	3705

TABLE 1. Numerical results.

$$F_{11}(t)=\int_0^t M_{12}(t-s)dF_{12}(s),\ F_{12}(t)=M_{12}(t),\ F_{13}(t)=\int_0^t M_{12}(t-s)dF_{23}(s),$$
$$F_{21}(t)=M_{21}(t)+\int_0^t M_{23}(t-s)dF_{31}(s),\ F_{22}(t)=\int_0^t M_{21}(t-s)dF_{12}(s)+\int_0^t M_{23}(t-s)dF_{32}(s),$$
$$F_{23}(t)=M_{23}(t)+\int_0^t M_{21}(t-s)dF_{13}(s),\ F_{31}(t)=M_{31}(t),\ F_{32}(t)=\int_0^t M_{31}(t-s)dF_{12}(s),$$
$$F_{33}(t)=\int_0^t M_{31}(t-s)dF_{13}(s).$$

Therefore, the equation (9) can be written as

$$\begin{aligned}
&\alpha_0^{ij}\int_0^T T_l(t)T_0(t)dt+\alpha_1^{ij}\int_0^T T_l(t)T_1(t)dt+\alpha_2^{ij}\int_0^T T_l(t)T_2(t)dt=\\
&=\delta_{ij}\int_0^T T_l(t)[1-H_i(t)]dt+\alpha_0^{1j}\int_0^T T_l(t)\int_0^t T_0(s)dM_{i1}(t-s)dt\\
&+\alpha_0^{2j}\int_0^T T_l(t)\int_0^t T_0(s)dM_{i2}(t-s)dt+\alpha_0^{3j}\int_0^T T_l(t)\int_0^t T_0(s)dM_{i3}(t-s)dt\\
&+\alpha_1^{1j}\int_0^T T_l(t)\int_0^t T_1(s)dM_{i1}(t-s)dt+\alpha_1^{2j}\int_0^T T_l(t)\int_0^t T_1(s)dM_{i2}(t-s)dt\\
&+\alpha_1^{3j}\int_0^T T_l(t)\int_0^t T_1(s)dM_{i3}(t-s)dt+\alpha_2^{1j}\int_0^T T_l(t)\int_0^t T_2(s)dM_{i1}(t-s)dt\\
&+\alpha_2^{2j}\int_0^T T_l(t)\int_0^t T_2(s)dM_{i2}(t-s)dt+\alpha_2^{3j}\int_0^T T_l(t)\int_0^t T_2(s)dM_{i3}(t-s)dt,
\end{aligned}$$

for $0\le i,j\le n-1$ and $0\le l\le n$, i.e. a linear system of equations having a sparse matrix, mainly due to the local support property of the trigonometric B-spline functions.

In order to obtain the reliability of the system modeled by the above kernel matrix, the following system of equations have to be solved

$$\begin{cases} F_{13}(t)=\int_0^t M_{12}(t-s)dF_{23}(s),\\ F_{23}(t)=M_{23}(t)+\int_0^t M_{21}(t-s)dF_{13}(s), \end{cases}$$

which can also be solved in a Galerkin framework.

4. Numerical results and concluding remarks

Let us consider a system having 3 states: it is full operational (state 1), it is partially operational, but available (state 2), and is is under repair and unavailable (state 3). The kernel matrix $M(t)$ considered above has the following elements: $M_{12}(t)=\mathrm{P}\{$the system partially fails before or at time t$\}=1-\exp(-\lambda t)$, $M_{21}(t)=\mathrm{P}\{$the system is completely recovered before or at time t and $t<L\ \}=1-\exp(-\mu t)-(\exp(-\mu L)-\exp(-\mu t))$ if $t\ge L$, and $M_{21}(t)=1-\exp(-\mu t)$ if $t<L$; $M_{23}(t)=\mathrm{P}\{$the

system is not completely recovered before or at time t and $t \geq L$ } $= 0$ if $t < L$, but $1 - \exp(-\mu L)$ if $t \geq L$; $M_{31}(t) = \mathrm{P}\{$ the system is partially or completely recovered before or at time t$\} = 1 - \exp(-\mu t)$.

For $\lambda = 0.002083(hours^{-1}), \mu = 0,04167(hours^{-1})$, $L = 2$ (corresponding at 48 hours), $n = 40$ and $v = 2$, using the trigonometric Simpson quadrature method [7], the results presented in Table 1 are obtained.

Comparing the above results against the approach given in [11], less accurate results are obtained by using the Galerkin approach. Another disadvantage consists of its computational effort required to estimate the coefficients. Numerical quadrature methods have to be used to compute the coefficients, and a large sparse linear system of equations have to be solved. Gauss or Quasi-Gauss-Newton methods [3] can be used to solve the linear systems which appear in the modeling process.

However, this method works for general kernel matrix. This means that a large variety of transitions M_{ij} can be used, in closed form (see [8] for a generalization of some classical models) or estimated from data failures and/or data maintenance reports.

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Matching theorems and applications

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Abstract. We obtain generalizations of Fan's matching theorems for open or closed covering related to an acyclic map. Finally, applications concerning coincidence theorems and section results are given.

Keywords: acyclic map, convex space, matching theorem, coincidence theorem

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1. Introduction and preliminaries

In this paper, using the Eilenberg and Montgomery fixed point theorem [4], a matching theorem involving acyclic maps is obtained . Further on we establish new KKM theorems, mutually equivalent with another matching theorems. In the last section some applications concerning coincidence and section theorems are given.

Let us recall some concepts and notations. A *convex space* [10] is a nonempty convex set in a vector space with an arbitrary topology that induces the Euclidean topology on the convex hulls of its finite subsets. A subset A of a topological space Y is said to be *compactly open* (resp. *closed*) in Y if for every compact set $K \subset Y$ the set $A \cap K$ is open (resp. closed) in K. A nonempty topological space is *acyclic* if all its reduced Čech homology groups over rational vanish.

A map $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y, that is, a function with the values $F(x) \subset Y$ for $x \in X$, and the fibers $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for $y \in Y$. If $A \subset X$, let be $F(A) = \cup\{F(x) : x \in A\}$.

For topological spaces X and Y a map $F : X \multimap Y$ is *upper semicontinuous* if the set $\{x \in X : F(x) \cap Y_1 \neq \emptyset\}$ is closed in X for each closed set Y_1 in Y. An upper semicontinuous map with compact acyclic values is called an *acyclic map*. A map $F : X \multimap Y$ is said to be *compact* if the range $F(X)$ is contained in a compact set of the topological space Y.

Throughout this paper the topological spaces will be supposed Hausdorff. For a set D let $\langle D\rangle$ denote the set of all nonempty finite subsets of X.

2. Matching theorems and KKM theorems

We start with the following fixed point result (see [4]).

Lemma 1. *Let Δ be an n-simplex with the Euclidean topology and let Y be a compact space. Let $F : \Delta \multimap Y$ be an acyclic map and let $g : Y \to \Delta$ be a continuous function. Then there is $x_0 \in \Delta$ such that $x_0 \in g(F(x_0))$ (or, equivalently, there are $x_0 \in \Delta$ and $y_0 \in Y$ such that $y_0 \in F(x_0)$ and $x_0 \in g(y_0)$).*

The following result generalizes Theorem 1 in [11] which in turn extends the open version of Fan's matching theorem [6].

Theorem 2. *Let D be a nonempty subset of a convex space, let Y be a topological space and let $G : D \multimap Y$ be a map such that:*

(i) *for each $x \in D$, $G(x)$ is compactly open in Y;*
(ii) $G(D) = Y$.

Then for each compact acyclic map $F : coD \multimap Y$ there exists $A \in \langle D\rangle$ such that $F(coA) \cap \cap\{G(x) : x \in A\} \neq \emptyset$.

Proof. Since F is a compact map, we may assume that Y is a compact space and, for each $x \in D$, $G(x)$ is an open subset of Y. Consequently, there is a set $D_1 = \{x_1, \ldots, x_n\} \in \langle D \rangle$ such that $Y = \cup_{i=1}^n G(x_i)$. Let $\{\alpha_i\}_{i=1}^n$ be a continuous partition of unity subordinated to this covering of Y. Define the continuous function $g : Y \to \mathrm{co}D_1$, $G(y) = \sum_{i=1}^n \alpha_i(y)x_i$. By Lemma 1 there are $x_0 \in \mathrm{co}D_1$ and $y_0 \in Y$ such that $x_0 = g(y_0)$ and $y_0 \in F(x_0)$. Denote $I = \{i \in \{1, \ldots, n\} : \alpha_i(y_0) > 0\}$. Obviously $I \neq \emptyset$. If $i \in I$, then y_0 lies in the support of α_i and, therefore, in $G(x_i)$. Thus $y_0 \in \cap\{G(x_i) : i \in I\}$. On the other hand, $x_0 = g(y_0) \in \mathrm{co}\{x_i : i \in I\}$, whence $y_0 \in F(x_0) \subset F(\mathrm{co}\{x_i : i \in I\})$.

Taking $A = \{x_i : i \in I\}$ we get $y_0 \in F(\mathrm{co}A) \cap \cap\{G(x) : x \in A\}$. □

Theorem 2 can be restated in its contraposition from and in terms of the complements $S(x)$ of $G(x)$ in Y as follows.

Theorem 3. *Let D be a nonempty subset of a convex space, let Y be a topological space and let $S : D \multimap Y$ be a map with compactly closed values. If there exists a compact acyclic map $F : coD \multimap Y$ such that*

$$F(coA) \subset S(A) \text{ for each } A \in \langle D \rangle,$$

then $\cap\{S(x) : x \in D\} \neq \emptyset$.

The above KKM theorem includes earlier results of Lassonde [9], Chang [3], Shioji [15]. The compactness condition imposed to the map F can be relaxed as in the next theorem.

Theorem 4. *Let D be a nonempty subset of a convex space, let Y be a topological space, let $S : D \multimap Y$ be a map and let $F : coD \multimap Y$ be an acyclic map such that:*

(i) *for each $x \in D$, $S(x)$ is compactly closed in Y;*
(ii) *for each $A \in \langle D \rangle$, $F(coA) \subset S(A)$;*
(iii) *there exists a nonempty compact subset K of Y such that either*
  (a) *$\cap\{S(x) : x \in A_0\} \subset K$, for some $A_0 \in \langle D \rangle$; or*
  (b) *for each $A \in \langle D \rangle$ there exists a compact convex subset L_A of coD, containing A such that*

$$F(L_A) \cap \cap\{S(x) : x \in L_A \cap D\} \subset K.$$

Then $\overline{F(coD)} \cap K \cap \cap\{S(x) : x \in D\} \neq \emptyset$.

Proof. Suppose the conclusion does not hold and put $G(x) = Y \setminus S(x)$, for $x \in D$. Since $\overline{F(\mathrm{co}D)} \cap K$ is compact and $G(x)$ is compactly open for each $x \in X$, there exists $A_1 \in \langle D \rangle$ such that

$$\overline{F(\mathrm{co}D)} \cap K \subset G(A_1). \tag{1}$$

We examine successively the two cases having in view to obtain a contradiction.

Case (a). In this case

$$\overline{F(\mathrm{co}D)} \cap K \subset Y \setminus K \subset G(A_0), \tag{2}$$

hence, by (1) and (2), $\overline{F(\mathrm{co}D)} \subset G(A)$, where $A = A_0 \cup A_1$. Since co$A$ is compact and F is upper semicontinuous compact valued, $F(\mathrm{co}A)$ is a compact set and $F(\mathrm{co}A) \subset G(A)$.

By Theorem 2 there exists a nonempty set $B \subset A$ such that

$$F(\mathrm{co}B) \cap \cap\{G(x) : x \in B\} \neq \emptyset, \text{ that is } F(\mathrm{co}B) \not\subset S(B).$$

This contradicts (ii).

Case (b). By hypothesis, there exists a compact convex set L such that $A_1 \subset L \subset \mathrm{co}D$ and

$$F(L) \cap \cap\{S(x) : x \in L \cap D\} \subset K \tag{3}$$

We claim that $F(L) \subset G(L \cap D)$. Taking into account (1) we have $F(L) \setminus K \subset G(L \cap D)$. Hence, $F(L) \subset G(L \cap D)$. Since $F(L)$ is compact, there exists $B \in \langle D \rangle$ such that $F(\mathrm{co}B) \subset F(L) \subset G(B)$. For the remainder of the proof we can follow that from Case (a). □

Theorem 4 is a slight generalization of Theorem 3 in [13], which, in turn, generalizes earlier results of Fan [5],[6], Lassonde [9].

Theorem 4 can be also stated in its contraposition from and in terms of the complements $G(x)$ of $S(x)$. In this way the following generalization of Theorem 2 is obtained.

Theorem 5. *Let D be a nonempty subset of a convex space, let Y be a topological space, let $G : D \multimap Y$ be a map and let $F : coD \multimap Y$ be an acyclic map. Suppose that:*

(i) *for each $x \in D$, $G(x)$ is compactly open in Y;*
(ii) *there exists a nonempty compact subset K of Y such that $\overline{F(coD)} \cap K \subset G(D)$;*
(iii) *either*
  (a) *$Y \setminus K \subset G(A_0)$ for some $A_0 \in \langle D \rangle$; or*
  (b) *for each $A \in \langle D \rangle$ there exists a compact convex subset L_A of coD, containing A such that $F(L_A) \setminus K \subset G(L_A \cap D)$.*

Then there exists an $A \in \langle D \rangle$ such that $F(coA) \cap \cap\{S(x) : x \in A\} \neq \emptyset$.

The following lemma is necessary in order to obtain an open-valued version of Theorem 3. Its proof can be found in [1].

Lemma 6. *Let D be a nonempty finite subset of a convex space, let Y be a compact space, let $G : D \multimap Y$ be an open-valued map and let $F : coD \multimap Y$ be an acyclic map such that:*

$$F(coA) \subset G(A) \text{ for each nonempty set } A \subset D.$$

Then there is a closed-valued map $S : D \multimap Y$ such that $S(x) \subset G(x)$ for all $x \in D$ and $F(coA) \subset S(A)$ for each nonempty set $A \subset D$.

Theorem 7. *Let D be a nonempty subset of a convex space, let Y be a topological space, let $G : D \multimap Y$ be a map with compactly open values. If there exists an acyclic map $F : coD \multimap Y$ such that $F(coA) \subset G(A)$ for each $A \in \langle D \rangle$, then $\{G(x) : x \in D\}$ has the finite intersection property.*

Proof. Let $D_1 \in \langle D \rangle$. Since F is an upper semicontinuous compact-valued map, $Y_1 = F(\mathrm{co}D)$ is a compact set. By Lemma 6, there is a closed-valued map $S : D \multimap Y_1$ such that $S(x) \subset G(x) \cap Y_1$ for all $x \in D_1$ and $F(\mathrm{co}A) \subset S(A)$ for each $A \in \langle D_1 \rangle$. According to Theorem 3, we have $\cap\{G(x) \cap Y_1 : x \in D_1\} \supset \cap\{S(x) : x \in D_1\} \neq \emptyset$. □

The origin of Theorem 7 is due to Kim [8]. Our theorem includes earlier results of Lassonde [10] and Park [12, 14].

In turn, Theorem 7 can be easily reformulated obtaining the following matching theorem, which is a closed-valued version of Theorem 2.

Theorem 8. *Let D be a nonempty finite subset of a convex space, let Y be a topological space and let $S : D \multimap Y$ be a map such that:*

(i) *for each $x \in D$, $S(x)$ is compactly closed in Y;*
(ii) *$S(D) = Y$.*

Then for each acyclic map $F : coD \multimap Y$ there exists $A \in \langle D \rangle$ such that $F(coA) \cap \cap\{S(x) : x \in A\} \neq \emptyset$.

3. Coincidence theorems and applications

As application of Theorem 5 we give the following coincidence theorem.

Theorem 9. *Let D be a nonempty subset of a convex space, let Y be a topological space, let $G : D \multimap Y$, $T : coD \multimap Y$ be two maps and let $F : coD \multimap Y$ be an acyclic map. Suppose that conditions (i)-(iii) in Theorem 5 hold. Moreover assume that*

(iv) *for each $y \in F(coD)$, $co(G^{-1}(y)) \subset T^{-1}(y)$.*

Then there exists $x_0 \in coD$ such that $F(x_0) \cap T(x_0) \neq \emptyset$.

Proof. By Theorem 5, there exist $A \in \langle D\rangle$ and $y_0 \in F(\mathrm{co}A) \cap \cap\{G(x) : x \in A\}$. Therefore, $y_0 \in F(x_0)$ for some $x_0 \in \mathrm{co}A$.

On the other hand, from $y_0 \in \cap\{G(x) : x \in A\}$, taking into account (iv), we get $x_0 \in \mathrm{co}A \subset \mathrm{co}(G^{-1}(y_0)) \subset T^{-1}(y_0)$. Consequently, $y_0 \in F(x_0) \cap T(x_0)$. □

Theorem 9 extends results of Tarafdar [17, 18], Ben-El-Mechaieck et al. [2], Park [11].

Similarly, using as argument Theorem 8 instead of Theorem 5, we can readily prove the following theorem:

Theorem 10. *Let D be a nonempty finite subset of a convex space, let Y be a topological space, let $S : D \multimap Y$, $T : coD \multimap Y$ be two maps and let $F : coD \multimap Y$ be an acyclic map. Suppose that conditions (i), (ii) in Theorem 8 hold. Moreover assume that:*

(iii) *for each $y \in F(coD)$, $co(S^{-1}(y)) \subset T^{-1}(y)$.*

Then there exists $x_0 \in coD$ such that $F(x_0) \cap T(x_0) \neq \emptyset$.

Using Theorems 9 and 10 we obtain two section theorems including earlier results due to Takahashi [16], Ha [7], Shioji [15].

Theorem 11. *Let D be a nonempty subset of a convex space, let Y be a topological space, let $F : coD \multimap Y$ be an acyclic map and let $\Omega \subset coD \times Y$, $\Gamma \subset D \times Y$ be two sets. Suppose that:*

(i) $\Gamma \subset \Omega$;

(ii) *for each $x \in coD$, $\{x\} \times F(x) \subset \Omega$;*

(iii) *for each $x \in D$, $\{y \in Y : (x, y) \in \Gamma\}$ is compactly closed in Y;*

(iv) *for each $y \in F(coD)$, $\{x \in coD : (x, y) \notin \Omega\}$ is convex;*

(v) *there exists a nonempty compact subset K of Y such that either*

(a) *for each $y \in Y \setminus K$, $A_0 \times \{y\} \not\subset \Gamma$, for some $A_0 \in \langle D\rangle$; or*

(b) *for each $A \in \langle D\rangle$ there is a compact convex subset L of coD, containing A such that $y \in F(L_A) \setminus K$, $(L_A \cap D) \times \{y\} \not\subset \Gamma$.*

Then there exists $y_0 \in \overline{F(coD)} \cap K$ such that $D \times \{y_0\} \subset \Gamma$.

Proof. Consider the maps $G : D \multimap Y$ and $T : \mathrm{co}D \multimap Y$ given by

$$G(x) = \{y \in T : (x, y) \notin \Gamma\} \text{ for } x \in D, \text{ and}$$

$$T(x) = \{y \in T : (x, y) \notin \Omega\} \text{ for } x \in \mathrm{co}D.$$

Suppose that the conclusion is false. Then $\overline{F(\mathrm{co}D)} \cap K \subset G(D)$. By (iii), for each $x \in D$, $G(x)$ is compactly open. Obviously, conditions (va), (vb) are equivalent to the conditions (iiia) and (iiib) by respectively in Theorem 5. By (iv), for each $y \in F(\mathrm{co}D)$, $T^{-1}(y)$ is convex, and taking into account (i) we infer that $\mathrm{co}(G^{-1}(y)) \subset T^{-1}(y)$.

Therefore all hypotheses of Theorem 9 are satisfied, hence T and F have a coincidence point $x_o \in \mathrm{co}D$. For $y \in T(x_0) \cap F(x_0)$ it follows $(x_0, y_0) \notin \Omega$ which contradicts (iii). □

In similar manner, from Theorem 10 we obtain

Theorem 12. *Let D be a finite nonempty subset of a convex space, let Y be a topological space, let $F : coD \multimap Y$ be an acyclic map, and let $\Omega \subset coD \times Y$, $\Gamma \subset D \times Y$ be two sets. Suppose that the conditions (i), (ii), (iv) in Theorem 11 hold. Moreover assume that*

(iii′) *for each $x \in D$, $\{y \in Y : (x, y) \in \Gamma\}$ is compactly open in Y.*

Then there exists $y_0 \in Y$ such that $D \times \{y_0\} \subset \Gamma$.

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Multi-time dynamics of paths in first-order jet spaces

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Abstract. Within the framework of jet spaces endowed with non-linear connection, both the special curves of the spaces (h-paths, v-paths and geodesics) and the Lorentz-type paths for Lagrangian-induced structures which extend the Riemannian classical electromagnetic field model are characterized. Several remarkable special cases are discussed and computer-drawn graphic Maple-V plots for paths are provided.

Keywords: Lagrangian, jet space, nonlinear connection, Cartan connection, deflection tensor, electromagnetic tensor, stationary curve, path, Lorentz equations.

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1. Basic geometrical objects on $J^1(T,M)$

The geometrized framework on osculating first and higher-order osculating spaces was introduced and widely studied by R.Miron and collaborators ([3], [4]). As a complementary extension of the tangent (first-order osculating) framework, during the last decade the geometric approach on first-order jet spaces ([7], [6], [1], [2]) was developed with significant results.

In the sequel let $\xi = (E = J^1(T,M), \pi, T \times M)$ be the first order jet bundle of the mappings $\varphi : T \to M$, where T and M are $\mathcal{C}^\infty$ real differentiable manifolds ($\dim T = m$, $\dim M = n$). The local jet coordinates on E will be denoted by

$$(t^\alpha, x^i, y^A)_{(\alpha,i,A)\in I_*} \equiv (y^\mu)_{\mu\in I},$$

where the set of indices I splits as follows

$$I = I_h \cup I_v, \quad I_h = I_{h_1} \cup I_{h_2}, \quad I_v = \overline{m+n+1, m+n+mn}$$
$$I_{h_1} = \overline{1,m}, \quad I_{h_2} = \overline{m+1, m+n}, \quad I_* = I_{h_1} \times I_{h_2} \times I_v$$

and the indices implicitly take values as described below:

$$\alpha, \beta, \cdots \in I_{h_1}; \quad i, j, \cdots \in I_{h_2}; \quad A, B, \cdots \in I_v; \quad \lambda, \mu, \cdots \in I.$$

Moreover, when appropriate, any index $A = m + n + n(i - m - 1) + \alpha$, will be identified to $A \equiv \binom{i}{\alpha}$ and we assume that $y^A \equiv x^{\binom{i}{\alpha}} = \frac{\partial x^i}{\partial t^\alpha}$.

We endow E with the extended Lagrangian of electrodynamics ([6]) of the form

$$L(t,x,y) = \tilde{g}_{AB}(t,x,y)y^A y^B + U_A(t,x)y^A + \Phi(t,x), \tag{1}$$

where $\tilde{g}_{AB}$ is a nondegenerate tensor field, $U_A(t,x)$ is a 1-form on E and $\Phi(t,x)$ is a scalar function on $T \times M$.

Further consider the Kronecker case

$$\tilde{g}_{AB} \equiv \tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t,x)g_{ij}(t,x,y), \tag{2}$$

where $h_{\alpha\beta}$ and g_{ij} are non-degenerate tensor fields. The derived Euler-Lagrange equations evidentiate a spray, which, under certain restrictive conditions provides a *non-linear connection* $N = \{N^A_\mu\}_{\mu\in I_h, A\in I_v}$ on E, leading to a splitting [7, 4] $TE = HE \oplus VE$, where $VE = Ker\ \pi_*$. Similarly, N determines the local adapted basis of $\mathcal{X}(E)$ denoted by ([1], [2]) $\mathcal{B} = \{\delta_\alpha, \delta_i, \delta_A\}_{(\alpha,i,A)\in I_*} \equiv \{\delta_\mu\}_{\mu\in I}$, with $\partial_\alpha = \frac{\partial}{\partial t^\alpha}, \partial_i = \frac{\partial}{\partial x^i}$; then the dual basis of $\mathcal{B}$ writes $\mathcal{B}^* = \{\delta^\alpha, \delta^i, \delta^A\}_{(\alpha,i,A)\in I_*} \equiv \{\delta^\mu\}_{\mu\in I}$. The existence of Lagrangian-derived non-linear connections in the general Kronecker case [6] represents still an open

problem. However, in the following cases, where $\tilde{g}$ admits a particular Kronecker splitting, the problem is tractable.

Remarkable particular cases

I. The ARL (almost Riemann-Lagrange) jet case has $h_{\alpha\beta}(t)$ metric tensor field on T; the Lagrangian produces the canonical nonlinear connection $N=(N_\beta^{\binom{i}{\alpha}}, N_j^{\binom{i}{\alpha}})$ of coefficients

$$N_\beta^{\binom{i}{\alpha}} = -\left|\begin{matrix}\gamma\\ \alpha\beta\end{matrix}\right| y^{\binom{i}{\gamma}}, \quad N_j^{\binom{i}{\alpha}} = \left|\begin{matrix}i\\ jk\end{matrix}\right| y^{\binom{k}{\alpha}} + \tfrac{1}{4}g^{ik}(2\partial_\alpha g_{jk} + h_{\alpha\beta}U_{\binom{k}{\beta}j}) \tag{3}$$

Here we have denoted the h_2-curl of U by $U_{\binom{k}{\beta}j} = \delta_{[j}U_{\binom{k]}{\beta}}$, where $\tau_{[i\ldots j]} = \tau_{i\ldots j} - \tau_{j\ldots i}$ and $\tau_{\{i\ldots j\}} = \tau_{i\ldots j} + \tau_{j\ldots i}$. In this case we have

$$\tilde{g}_{AB} = \tfrac{1}{2}\dot{\partial}^2_{AB}L. \tag{4}$$

More particularly, in the ARLS (almost Riemann-Lagrange separated) jet case, g_{ij} is a metric tensor field on M.

Considering a non-linear connection $N=\{N_\alpha^A, N_i^A\}$ fixed on E, a linear connection $\nabla=\{L^\lambda_{\mu\nu}\}_{\lambda,\mu,\nu\in I}$ in E has the coefficients relative to the adapted basis provided by $\delta^\lambda(\nabla_{\delta_\nu}\delta_\mu) = L^\lambda_{\mu\nu}, \ \ \forall\lambda,\mu,\nu\in I$. These group in $3^3=27$ distinct subsets, according to the three sets of indices.

Let us endow E with a semi-Riemannian metric

$$G = \underbrace{h_{\alpha\beta}(t)dt^\alpha\otimes dt^\beta}_{h} + \underbrace{g_{ij}(t,x,y)dx^i\otimes dx^j}_{g} + \underbrace{\tilde{g}_{AB}(t,x,y)\delta y^A\otimes\delta y^B}_{\tilde{g}}, \tag{5}$$

with $\tilde{g}_{AB}\equiv\tilde{g}_{\binom{i}{\alpha}\binom{j}{\beta}} = h^{\alpha\beta}(t)g_{ij}(t,x,y)$. In this case *the Cartan linear connection* $\nabla\in\Gamma_*(N)$ ([7], [6]) has the four essential sets of coefficients

$$L^\alpha_{\beta\gamma} = \left|\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right|, \ \ L^i_{j\gamma} = \frac{1}{2}g^{ik}\delta_\gamma g_{kj}, \ \ L^i_{jk} = \left|\begin{matrix}i\\ jk\end{matrix}\right|, L^i_{j\binom{k}{\gamma}} = \frac{1}{2}g^{il}(\delta_{\binom{\{k}{\gamma}}g_{jl\}} - \delta_{\binom{l}{\gamma}}g_{jk}), \tag{6}$$

which provide the other 5 derived sets $L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma} = \delta^\beta_\alpha L^i_{j\gamma} - \delta^i_j\left|\begin{matrix}\beta\\ \alpha\gamma\end{matrix}\right|, \ \ L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k} = \delta^\beta_\alpha\left|\begin{matrix}i\\ jk\end{matrix}\right|, L^{\binom{i}{\alpha}}_{\binom{j}{\beta}C} = \delta^\beta_\alpha L^i_{jC}, L^\alpha_{\beta j} = 0, \quad L^\alpha_{\beta C} = 0.$

The Liouville field $\mathcal{C} = y^A\delta_A$ of (E,N) produces the *deflection tensor fields* $d^A_\mu = \delta^A\nabla_{\delta_\mu}\mathcal{C}$ ($\mu\in I, A\in I_v$), and further, the associated to N and ∇ electromagnetic 2-form $F = F_{A\mu}\delta y^A\wedge\delta y^\mu$, whose nontrivial components are

$$\begin{cases} F_{A\beta} \equiv F_{\binom{i}{\alpha}\beta} = \frac{1}{2}\left(h^{\alpha\gamma}g_{ik}y^{\binom{k}{\gamma}}\right)_{|\beta]}, \ \ F_{AB}\equiv F_{\binom{i}{\alpha}\binom{j}{\beta}} = \frac{1}{2}\tilde{g}_{\binom{[i}{\alpha}C}y^C{}_{|\binom{j]}{\beta}}, \\ F_{Aj}\equiv F_{\binom{i}{\alpha}j} = \frac{1}{2}\,d_{\binom{[i}{\alpha}j]} = \frac{1}{2}\,y_{\binom{[i}{\alpha}|j]} = \frac{1}{2}\left(y^{\binom{k}{\gamma}}h^{\alpha\gamma}g_{k[i}\right)_{|j]}, \end{cases} \tag{7}$$

where $|\alpha$, $|i$ and $|A$ denote the covariant derivations given by ∇_{δ_μ}, for $\mu\in I_{h_1}, I_{h_2}$ and I_v respectively. While the raising/lowering of the indices is performed by G, we canonically attach the electromagnetic force

$$\tilde{F} = F_A^\mu\delta_\mu\otimes\delta^A, \qquad F^\alpha_A = h^{\alpha\beta}F_{A\beta}, \ F^i_A = g^{ij}F_{Aj}, \ F^C_A = g^{CD}F_{AD}. \tag{8}$$

We note that in the ARLS case, the nontrivial coefficients of the Cartan connection are [6] $\{L^\alpha_{\beta\gamma} = \left|\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right|, \ L^i_{jk} = \left|\begin{matrix}i\\ jk\end{matrix}\right|\}$.

More particularly, in the ARLS case with $m=1, \ n=4$ and $h_{11}=1$, one finds as particular case, the pseudo-Riemannian weak gravitational model endowed with the metric $g_{ij}(x) = \eta_{ij}+\varepsilon_{ij}(x)$, where the weakness of the gravitational field g_{ij} is expressed by its decomposition into the flat Minkowski metric $n_{ij} = diag(-1,1,1,1)$ and a small perturbation $\varepsilon_{ij}(x)$, a symmetric tensor field with $|\varepsilon_{ij}(x)| << 1$.

2. Paths and Lorentz curves on $J^1(T,M)$

In the following we consider smooth curves $c: J=[a,b]\subset \mathbb{R}\to E$, having their images inside a chart $\tilde{U}\subset E$, locally given by

$$c(s)=(t^{\alpha}(s),x^i(s),y^A(s))\equiv (y^{\mu}(s)),\forall t\in J$$

and we fix a linear $N-$connection ∇ on E.

Definitions. a) The field $\mathcal{V}=\frac{\delta y^{\mu}}{\mathrm{d}s}$ defined on c is be called the *covariant velocity* field of the curve c. Its components are explicitely given by

$$\{\mathcal{V}^{\mu}\}_{\mu\in I}\equiv\left(\dot{t}^{\alpha},\dot{x}^i,\frac{\delta y^a}{\mathrm{d}s}=\dot{y}^A+N^A_{\beta}\dot{t}^{\beta}+N^A_j\dot{x}^j\right)_{(\alpha,i,A)\in I_*},$$

where the dot stands for the s-differentiation. We have also denoted $\mathcal{F}=\mathcal{F}^{\mu}\delta_{\mu}$, where

$$\mathcal{F}^{\mu}=\frac{\nabla\mathcal{V}^{\mu}}{ds}\overset{not}{=}\frac{\delta\mathcal{V}^{\mu}}{ds}+L^{\mu}_{\nu\rho}\mathcal{V}^{\nu}\mathcal{V}^{\rho}$$

is the *covariant force* on c, which provides the motion of the test-body along c.

b) We call c a *stationary curve* with respect to ∇ iff $\mathcal{F}=0$ along the curve.

c) The curve c is called a $h-$*curve*, if $\pi_v(\mathcal{V})=0$, and a $v-$*curve*, if $\pi_h(\mathcal{V})=0$, where by π_h and π_v we denoted the $h-$ and $v-$projectors of the canonic splitting induced by N respectively.

d) An $h-/v-$curve which satisfies the extra condition $\mathcal{F}=0$, is called an $h-/v-$*path*, respectively.

Analytically, these curves are described by the following

Theorem. *Let $c: J\subset\mathbb{R}\to E$ be a curve. Then the following hold:*

a) c is a $h-$curve iff

$$\mathcal{V}^A=0\ \Leftrightarrow\ \tfrac{\delta y^A}{ds}=0\ \Leftrightarrow\ \dot{y}^A+N^A_{\alpha}\dot{t}^{\alpha}+N^A_j\dot{x}^j=0,\ \ \forall A\in I_v; \tag{9}$$

b) c is a $v-$curve iff

$$\mathcal{V}^{\mu}=0,\ \forall\mu\in I_h\ \Leftrightarrow\ \tfrac{\delta y^{\mu}}{ds}=0,\forall\mu\in I_h\ \Leftrightarrow\ c(s)=(t_0,x_0,y(s)),s\in J; \tag{10}$$

c) c is an $h-$path ("stationary $h-$curve or "horizontal geodesic") iff besides (9) it satisfies $\frac{d\mathcal{V}^{\mu}}{ds}+L^{\mu}_{\nu\rho}\mathcal{V}^{\nu}\mathcal{V}^{\rho}=0,\ \forall\mu\in I_h$. Note that the implicit sum in the right term involves just horizontal index types.

d) c is a $v-$path ("stationary $v-$curve or "vertical geodesic") iff besides (10) it satisfies

$$\frac{\delta\mathcal{V}^A}{ds}+L^A_{BC}\mathcal{V}^B\mathcal{V}^C=0,\ \forall A\in I_v. \tag{11}$$

Note that the implicit sum in the right term involves just vertical index types. The proof is computational.

Consider the electromagnetic tensor fields in (7) and (8), the metric G in (5), a fixed nonlinear connection N, and the Cartan connection attached to G having the coefficients (6). Then *the Lorentz equations* attached to G, N and ∇ have the generic shape

$$G_{\nu\rho}\frac{\nabla\mathcal{V}^{\rho}}{ds}=\tilde{F}_{A\nu}\mathcal{V}^A\quad\Leftrightarrow\quad\frac{\nabla\mathcal{V}^{\mu}}{ds}=\mathcal{F}_A{}^{\mu}\mathcal{V}^A, \tag{12}$$

where $\mathcal{V}=\mathcal{V}^{\mu}\delta_{\mu}$ is *the covariant velocity* along the considered extended path of the moving test-particle. In detail, the Lorentz equations have the form [2]

$$\ddot{t}^{\alpha}+L^{\alpha}_{\beta C}\dot{t}^{\beta}\mathcal{V}^C+L^{\alpha}_{jC}\dot{x}^j\mathcal{V}^C+L^{\alpha}_{\beta\gamma}\dot{t}^{\beta}\dot{t}^{\gamma}+L^{\alpha}_{j\gamma}\dot{x}^j\dot{t}^{\gamma}+L^{\alpha}_{\beta k}\dot{t}^{\beta}\dot{x}^k+L^{\alpha}_{jk}\dot{x}^j\dot{x}^k=F^{\alpha}_B\mathcal{V}^B \tag{13}$$

$$\ddot{x}^i+L^i_{\beta C}\dot{t}^{\beta}\mathcal{V}^C+L^i_{jC}\dot{x}^j\mathcal{V}^C+L^i_{\beta\gamma}\dot{t}^{\beta}\dot{t}^{\gamma}+\qquad L^i_{j\gamma}\dot{x}^j\dot{t}^{\gamma}+L^i_{\beta k}\dot{t}^{\beta}\dot{x}^k+L^i_{jk}\dot{x}^j\dot{x}^k=F^i_B\mathcal{V}^B \tag{14}$$

$$\dot{\mathcal{V}}^A+N^A_{\alpha}\dot{t}^{\alpha}+N^A_i\dot{x}^i+L^A_{C\beta}\mathcal{V}^C\dot{t}^{\beta}+\qquad L^A_{Cj}\mathcal{V}^C\dot{x}^j+L^A_{BC}\mathcal{V}^B\mathcal{V}^C=F^A_B\mathcal{V}^B, \tag{15}$$

where $\mathcal{V}^A=\dot{y}^A+N^A_{\beta}\dot{t}^{\beta}+N^A_i\dot{x}^i,\ A\in I_v$.

a) The *Lorentz h-paths* satisfy the extra conditions $\mathcal{V}^A = 0,\ A \in I_v$ and since the right-hand side of (13)-(15) is identically vanishing, they *coincide with the usual h-paths of* (E, N, ∇).

b) The *Lorentz v-paths*, have fixed base-point $(t, x) = (t_0, x_0) \in T \times M$, i.e., $\mathcal{V}^\mu = 0,\ \mu \in I_h$, and, hence, the associated equations read

$$F_B^\alpha \mathcal{V}^B = 0, \quad F_B^i \mathcal{V}^B = 0, \quad F_B^A \mathcal{V}^B = \dot{\mathcal{V}}^A + L_{BC}^A \mathcal{V}^B \mathcal{V}^C.$$

In the ARLS case the electromagnetic tensors simplify to $F_A^\alpha \equiv F_{A\binom{i}{\beta}\gamma}^{\alpha} = F_A^B = 0$, $F_A^i = g^{ij}\tilde{F}_{Aj} = -\frac{1}{4}g^{ij}U_{Aj}$ and the nonvanishing Cartan essential coefficients become

$$L_{\beta\gamma}^\alpha = \left|{}_{\beta\gamma}^{\alpha}\right|, \quad L_{jk}^i = \left|{}_{jk}^{i}\right|, \quad L_{B\gamma}^A \equiv L_{B\binom{i}{\beta}\gamma}^{A\binom{i}{\alpha}} = -\delta_j^i \left|{}_{\alpha\gamma}^{\beta}\right|, \quad L_{Bk}^A \equiv L_{B\binom{i}{\beta}k}^{A\binom{i}{\alpha}} = -\delta_\alpha^\beta \left|{}_{jk}^{i}\right|.$$

Then the Lorentz equations (13)-(15) get the typical shape

$$\ddot{t}^\alpha + \left|{}_{\beta\gamma}^{\alpha}\right| \dot{t}^\beta \dot{t}^\gamma = 0, \quad \ddot{x}^i + \left|{}_{jk}^{i}\right| \dot{x}^j \dot{x}^k = -\frac{1}{4} g^{ij} U_{Aj} \mathcal{V}^A, \quad \dot{\mathcal{V}}^A = 0.$$

Note that in this case (g dependent on x only), the Berwald connection [7] has the same coefficients as the Cartan connection, and, hence, the associated Lorentz curves, h- and v-paths are described by the same equations. The Lorentz h-paths obey the extra equations $\dot{y}^A + N_\beta^A \dot{t}^\beta + N_j^A \dot{x}^j = 0$, which write explicitely as

$$\dot{y}^{\binom{i}{\alpha}} - \left|{}_{\alpha\beta}^{\gamma}\right| y^{\binom{i}{\gamma}} \dot{t}^\beta + \left(\left|{}_{jk}^{i}\right| y^{\binom{k}{\alpha}} + \frac{1}{4} g^{ik} h_{\alpha\beta} U_{\binom{k}{\beta}j} \right) \dot{x}^j = 0.$$

In the same way, the Lorentz v-paths for the Cartan connection satisfy the extra condition $-\mathcal{V}^A = 2\dot{\mathcal{V}}^A$, having as solutions flat rays within the fibers of E - in accordance with the particular case $J^1(\mathbb{R}, M) \equiv TM$ studied in [5].

In the ARLS uniparametric case, (for $m = 1$ and $s = t^1 = t$), we can use *the Finsler-Lagrange tangent space notations* from [4]. Shifting the indices left by one unit ($I_{h_2} = \overline{1,n},\ I_v = \overline{n+1,2n}$), we have $y^A \equiv y^{\binom{i}{1}} \overset{not}{=} y^i$, and set locally $h_{11} = 1$. For the Lagrangian (1) of particular form

$$L(x, y) = mc\,\gamma_{ij}(x) y^i y^j + \frac{2e}{m} U_i(x) y^i + \Phi(x), \tag{16}$$

with γ_{ij} pseudo-Riemannian metric and $U = U_i dx^i$ 1-form on M, the fundamental tensor derived from L via (4) is $\tilde{g}_{\binom{i}{1}\binom{j}{1}}(t, x, y) = g_{ij}(x) = mc\,\gamma_{ij}(x)$. The non-linear connection induced by L has the components $N_1^A = 0, N_j^{\binom{i}{1}} = \left|{}_{jk}^{i}\right| y^k + g^{ik} U_{\binom{k}{1}j}$, $i = \overline{1,n},\ A = \overline{n+1,2n}$, with $U_{\binom{k}{1}} = \frac{e}{m} A_k$. For ∇ Cartan connection, the Lorentz equations (14) confine to the known ones of Lagrange spaces [4], the equations of the Lagrangian spray $G^i = \frac{1}{2}\gamma_{jk}^i y^j y^k + \frac{e}{2m^2 c}\gamma^{ij} A_{[j;k]} y^k$ [3, p. 171]. Remark that in the absence of the electromagnetic force F_{μ_A}, the equations (12) become the equations of stationary curves of the connection ∇. In the absence of U, the equations (12) become *the equations of geodesics* of the manifold M and the equations of $h-paths$ become the Lorentz equations.

3. Numerical simulation with Maple V

In the ARLS uniparametric case consider $m = 1, T = \mathbb{R}, h_{11} = 1$ and $n = 2$. For M endowed with the Lagrangian (1) where $g = mc\gamma_{ij}$ with the potential $\bar{U}$ in L given by $\bar{U} = \varepsilon(x^1 dx^2 - x^2 dx^1)$, $\varepsilon \in \mathbb{R}$, for $a = \varepsilon e (m^2 c)^{-1}$, we have the appropriately rescaled Lorentz-type equations

$$\ddot{x}^i + \left|{}_{jk}^{i}\right| \dot{x}^j \dot{x}^k = (-1)^{i+1} a (g^{i1}\dot{x}^2 + g^{i2}\dot{x}^1),\ i = \overline{1,2}. \tag{17}$$

Further we exemplify the influence of F on h-paths for these three cases. Using Maple V programming, computer-drawn images were obtained representing the Lorentz-type curves for different values of the control parameter a. We note that, when the influence of the generalized electric potentials $U_i(x)$ disappears (for $a = 0$), the sheaves of geodesics of the manifold M (marked with thick lines) are

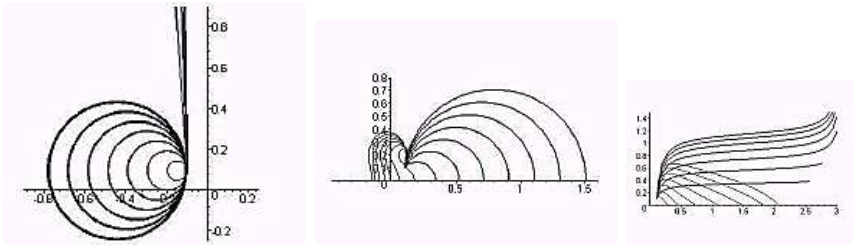

FIGURE 1

obtained; for nonvanishing values of a ($a = -512$ for Euclidean case, $a = -1024$ for the Poincaré half-plane, $a = 2$ for the sphere respectively), the solutions of the system (17) deform to Lorentz curves, under the influence of the electromagnetic tensor field, as seen below.

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Numerical method for Volterra integral equations

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Abstract. In this paper using the successive approximations method and a quadrature rule with the error estimation in terms of the third derivative we obtain a numerical method for solving Volterra nonlinear integral equations of second type . Afterward, we present a suitable algorithm for this method, illustrated by a numerical example.

Keywords: Volterra integral equations, successive approximations method, perturbed trapezoidal quadrature rule.

2000 Mathematics Subject Classification: 45D05, 65R20

1. Introduction

In this paper we present a numerical method for Volterra nonlinear integral equations of second type. Under the hypotheses of the local existence and uniqueness theorem we use the successive approximations method to approximate the solution of the Volterra integral equation

$$y(x) = \int_a^x H(s, y(s))ds + f(x),\ x \in [a, b] \tag{1}$$

In order to compute the integrals involved in the steps of the successive approximation method we use a perturbed trapezoidal quadrature rule with uniform distributed knots and the remainder in terms of the third derivative. In this way, we obtain a numerical method and give the algoritm of this method illustrated by an exemple.

The numerical methods for integral equations use quadrature formulas. Likewise, for linear integral equations quadrature formulas based on Riemann's sums as in [9] or rectangular and trapezoidal quadrature rules as in [7] are used to obtain a triangular algebraic linear system. The use of the successive approximations method is suitable to nonliniar integral equations. In [6] D.V.Ionescu obtains numerical methods for ordinary differential equations using the successive approximations. Obviously, this method can be adapted for integral equations. In [3] the method in [6] is extended to nonlinear integral equations using the trapezoidal quadrature formula. In [4] C. Iancu obtains a numerical method for the delay integral equation which arise in biomathematics, $x(t) = \int_{t-z}^{t} f(s, x(s))ds,\ t \in [0, T]$, using the trapezoidal quadrature formula while in [2] a numerical method using a perturbed trapezoidal quadrature rule with the remainder in terms of the third derivative is obtained. Here,we use and adapt the quadrature rule from [2] to approximate the solution of (1).

2. Existence and uniqueness of the solution

In $C[a, b]$ we consider the Bielecki norm, $\| u \|_b = \max_{x\in[a,b]} |u(x)|e^{-z(x-a)}, \forall\ u \in C[a, b]$, with a convenient constant $z > 0$. According to [3] we denote by $B[a, b]$ the Banach space $(C[a, b], \| \cdot \|_b)$.

On a closed ball $\overline{B}(f, R) \subset B[a, b]$ we define the mapping, $A : \overline{B}(f, R) \to B[a, b]$,

$$A(y)(x) = \int_a^x H(s, y(s))ds + f(x), \forall\ \ x \in [a, b], \forall\ y \in \overline{B}(f, R).$$

As in [3] we obtain the result:

Theorem 1. *Let H and F be two real functions satisfying the following conditions:*

(i) $\exists\ r > 0,\ \exists\ M, m \geq 0$ *such that* $(b-a)M + m \leq r$ *and* $|f(x)| \leq m,\ |H(x,y)| \leq M, \forall x \in [a,b], \forall y \in [-r,r]$;
(ii) $H \in C([a,b] \times [-r.r]), f \in C[a,b]$;
(iii) $(\exists) L > 0$ *such that* , $|H(x,u) - H(x,v)| \leq L|u-v|, \forall x \in [a,b], \forall u, v \in [-r,r]$;
(iv) $(\exists) R > 0$ *such that* $M(b-a) < R$.

Then for any $z > 0$ *such that* $\frac{L}{z} < 1$ *equation (1) has in* $\overline{B}(f,R)$ *a unique solution which can be obtained by the method of successive approximations.*

Let us construct the sequence of successive approximations, given by $\varphi_m = A(\varphi_{m-1}), \forall m \in \mathbb{N}^*$:

$$
\begin{aligned}
&\varphi_0(x) = f(x), x \in [a,b] \\
&\varphi_1(x) = \textstyle\int_a^x H(s, \varphi_0(s))ds + f(x) = \int_a^x H(s, f(s))ds + f(x), x \in [a,b] \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\varphi_m(x) = \textstyle\int_a^x H(s, \varphi_{m-1}(s))ds + f(x), x \in [a,b], \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{2}
$$

If we denote by φ the unique solution of (1), according to [3], we have the estimation $\|\varphi - \varphi_m\|_B \leq \frac{(\frac{L}{z})^m}{1-\frac{L}{z}}\|\varphi_0 - \varphi_1\|_B, \forall m \in \mathbb{N}^*$. Taking $z = 10L$, we obtain the following estimation $\|\varphi - \varphi_m\|_B \leq \frac{R}{9 \cdot 10^{m-1}}$, $\forall m \in N^*$, since $\|\varphi_0 - \varphi_1\|_B \leq \max\limits_{x \in [a,b]} \int_a^x |H(s, f(s))|ds \leq M(b-a)$. Consequently, using the relation between the Tchebytshev's and Bielecki's norms, we obtain, according to condition (iv), the estimation

$$
|\varphi(x) - \varphi_m(x)| \leq \|\varphi - \varphi_m\|_B \cdot e^{z(b-a)} \leq \frac{R \cdot e^{10L(b-a)}}{9 \cdot 10^{m-1}}, \forall x \in [a,b], \forall m \in \mathbb{N}. \tag{3}
$$

3. Numerical method

In order to compute the integrals from (2) we use a perturbed trapeizoidal quadrature rule. The following quadrature formula was first proposed by N. Obreschkoff [8] in 1940 , (see also [5])

$$
\int_a^b f(x)dx = \frac{(b-a)}{2}[f(a) + f(b)] - \frac{(b-a)^2}{12}[f'(b) - f'(a)] + R(f),
$$

and he obtained the remainder estimation (according to [5] page 55), $|R(f)| \leq \frac{(b-a)^5}{720}\|f^{\ iv}\|$. Recently , in [1] , for the above quadrature formula, a better remainder estimation which require only the third derivative: $|R(f)| \leq \frac{(b-a)^4}{160}\|f'''\|$ was obtained.

Using a uniform partition of $[a,b]$, $\Delta : a = x_0 < x_1 < \ldots < x_n = b$, $x_i = a + \frac{i(b-a)}{n}, \forall i = \overline{0,n}$, in [1] the following quadrature rule

$$
\int_a^b f(x)dx = \frac{(b-a)}{2n}[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)] - \frac{(b-a)^2}{12n^2}[f^{'}(b) - f^{'}(a)] + R_n(f), \tag{4}
$$

was obtained with the remainder estimation $|R_n(f)| \leq \frac{(b-a)^4}{160n^3} \cdot \|f'''\|$, which holds if $f \in C^3[a,b]$.

We apply the quadrature rule (4) to compute the integrals from (2). For this reason we consider the function $F : [a,b] \to \mathbb{R}$, $F(x) = H(x, \varphi(x))$. If $H \in C^3([a,b] \times [-r,r])$ and $f \in C^3[a,b]$, $H \in C^3([a,b] \times [-r,r]), f \in C^3[a,b]$,it follows that $\varphi \in C^3[a,b]$ and

$$
\begin{aligned}
&F^{\ '''}(x) = \tfrac{\partial^3 H}{\partial x^3}(x, \varphi(x)) + 3\tfrac{\partial H^3}{\partial x^2 \partial y}(x, \varphi(x)) \cdot \varphi'(x) + 3\tfrac{\partial^3 H}{\partial x \partial y^2}(x, \varphi(x)) \cdot [\varphi'(x)]^2 + \\
&+ \tfrac{\partial^3 H}{\partial y^3}(x, \varphi(x)) \cdot [\varphi'(x)]^3 + 3\tfrac{\partial^2 H}{\partial x \partial y}(x, \varphi(x)) \cdot \varphi''(x) + 3\tfrac{\partial^2 H}{\partial y^2}(x, \varphi(x)) \cdot \varphi'(x) \cdot \\
&\cdot \varphi''(x) + \tfrac{\partial H}{\partial y}(x, \varphi(x)) \cdot \varphi'''(x), \forall x \in [a,b].
\end{aligned}
$$

Consequently, we have,

$$
\begin{aligned}
&\|F'''\| \leq M_3(1 + M + D_1)^3 + 4M_1M_2(1 + M + D_1)^2 + \\
&+ (1 + M + D_1)(M_1{}^3 + 3M_2D_2) + M_1{}^2D_2 + M_1D_3 = M'''
\end{aligned}
$$

where $M=\max\{H(x,y):x\in[a,b],y\in[-r,r]\}$,

$$M_1=\max\{\|\tfrac{\partial H}{\partial x}\|,\|\tfrac{\partial H}{\partial y}\|\},\ M_2=\max\{\|\tfrac{\partial^2 H}{\partial x^2}\|,\|\tfrac{\partial^2 H}{\partial x\partial y}\|,\|\tfrac{\partial^2 H}{\partial y^2}\|\}$$
$$M_3=\max\{\|\tfrac{\partial^3 H}{\partial x^3}\|,\|\tfrac{\partial^3 H}{\partial x\partial y}\|,\|\tfrac{\partial^3 H}{\partial x\partial y^2}\|,\|\tfrac{\partial^3 H}{\partial y^3}\|\}$$
$$\|\tfrac{\partial^\alpha H}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\|=\max\{|\tfrac{\partial^{|\alpha|}H(x,y)}{\partial x^{\alpha_1}\partial y^{\alpha_2}}|:x\in[a,b],y\in[-r,r],\alpha_1+\alpha_2=|\alpha|\}$$

$D_i=\|f^{(i)}\|=\max\{|f^{(i)}(x)|:x\in[a,b]\}$, for $i\in\{1,2,3\}$, since $\|\varphi'\|\le M+D_1$, $\|\varphi''\|\le M_1(1+M+D_1)+D_2$ and $\|\varphi'''\|\le M_2(1+M+D_1)^2+{M_1}^2(1+M+D_1)+M_1D_2+D_3$.

Starting with the relation, $\varphi_m(x)=\int_a^x H(s,\varphi_{m-1}(s))ds+f(x)$, for every $x\in[a,b]$ and using the quadrature rule (4) with equidistant knots, we have

$$\begin{aligned}&\varphi_m(x_k)=\tfrac{(b-a)}{2n}[H(a,\varphi_{m-1}(a))+2\sum_{i=1}^{k-1}H(x_i,\varphi_{m-1}(x_i))+H(x_k,\varphi_{m-1}(x_k))]+\\&+f(x_k)-\tfrac{(b-a)^2}{12n^2}[\tfrac{\partial H}{\partial x}(x_k,\varphi_{m-1}(x_k))+\tfrac{\partial H}{\partial y}(x_k,\varphi_{m-1}(x_k))\cdot(H(x_k,\varphi_{m-2}(x_k))+\\&f\ '(x_k))-\tfrac{\partial H}{\partial x}(a,\varphi_{m-1}(a))-\tfrac{\partial H}{\partial y}(a,\varphi_{m-1}(a))(H(a,\varphi_{m-1}(a))+f'(a))]+R_{m,k}\end{aligned}\tag{5}$$

$\forall k=\overline{1,n},\forall m\in\mathbb{N},m\ge 2$, where, $\varphi_0(x_k)=f(x_k),\forall k=\overline{1,n}$, and $\varphi_1(x_k)=$
$=\tfrac{(b-a)}{2n}[H(a,f(a))+2\sum_{i=1}^{k-1}H(x_i,f(x_i))+H(x_k,f(x_k))]-\tfrac{(b-a)^2}{12n^2}\cdot[\tfrac{\partial H}{\partial x}(x_k,f(x_k))+$
$+\tfrac{\partial H}{\partial y}(x_k,f(x_k))f\ '(x_k)-\tfrac{\partial H}{\partial x}(a,f(a))-\tfrac{\partial H}{\partial y}(a,f(a))f\ '(a)]+R_{1,k}=\widetilde{\varphi}_1(x_k)+R_{1,k}$.

For the remainder $R_{m,k}$ we have the estimation: $|R_{m,k}|\le\tfrac{(b-a)^4}{160n^3}\cdot\|F'''\|\le\tfrac{(b-a)^4M'''}{160n^3},\forall m\in\mathbb{N}^*,\forall k=\overline{1,n}$, which does not depend on m or k.

4. The algorithm

Further by using the succesive approximations (2) and the formulas (5), we present an algorithm to approximate the solution φ of (1). So, we have

$$\begin{aligned}&\varphi_2(x_k)=\tfrac{(b-a)}{2n}[H(a,f(a))+2\sum_{i=1}^{k-1}H(x_i,\widetilde{\varphi_1}(x_i)+R_{1,i})+H(x_k,\widetilde{\varphi_1}(x_k)+R_{1,k})]\\&-\tfrac{(b-a)^2}{12n^2}\cdot[\tfrac{\partial H}{\partial x}(x_k,\widetilde{\varphi_1}(x_k)+R_{1,k})+\tfrac{\partial H}{\partial y}(x_k,\widetilde{\varphi_1}(x_k)+R_{1,k})\cdot[H(x_k,f(x_k))+\\&f'(x_k)]-\tfrac{\partial H}{\partial x}(a,f(a))-\tfrac{\partial H}{\partial y}(a,f(a))\cdot(H(a,f(a))+f'(a))]+f(x_k)+R_{2,k}\\&=\tfrac{(b-a)}{2n}\cdot[H(a,f(a))+2\sum_{i=1}^{k-1}H(x_i,\widetilde{\varphi_1}(x_i))+H(x_k,\widetilde{\varphi_1}(x_k))]-\tfrac{(b-a)^2}{12n^2}\cdot\\&[\tfrac{\partial H}{\partial x}(x_k,\widetilde{\varphi}_1(x_k))+\tfrac{\partial H}{\partial y}(x_k,\widetilde{\varphi_1}(x_k))\cdot(H(x_k,f(x_k))+f'(x_k))-\tfrac{\partial H}{\partial x}(a,f(a))-\\&\tfrac{\partial H}{\partial y}(a,f(a))\cdot(H(a,f(a))+f'(a))]+f(x_k)+\widetilde{R}_{2,k}=\widetilde{\varphi_2}(x_k)+\widetilde{R}_{2,k},\forall k=\overline{1,n}.\end{aligned}$$

By induction, for $m\in\mathbb{N},\ m\ge 3$ we obtain

$$\begin{aligned}&\varphi_m(x_k)=\tfrac{(b-a)}{2n}[H(a,f(a))+2\sum_{i=1}^{k-1}H(x_i,\widetilde{\varphi}_{m-1}(x_i)+R_{1,i})+H(x_k,\widetilde{\varphi}_{m-1}(x_k)+\\&+\widetilde{R}_{m,k})]-\tfrac{(b-a)^2}{12n^2}[\tfrac{\partial H}{\partial x}(x_k,\widetilde{\varphi}_{m-1}(x_k)+\widetilde{R}_{m-1,k})+\tfrac{\partial H}{\partial y}(x_k,\widetilde{\varphi}_{m-1}(x_k)+\widetilde{R}_{m-1,k})\cdot\\&\cdot[H(x_k,\widetilde{\varphi}_{m-2}(x_k)+\widetilde{R}_{m-2,k})+f\ '(x_k)]-\tfrac{\partial H}{\partial x}(a,f(a))-\tfrac{\partial H}{\partial y}(a,f(a))\cdot\\&\cdot(H(a,f(a))+f'(a))]+f(x_k)+R_{m,k}=\tfrac{(b-a)}{2n}\cdot[H(a,f(a))+\\&+2\sum_{i=1}^{k-1}H(x_i,\widetilde{\varphi}_{m-1}(x_i))+H(x_k,\widetilde{\varphi}_{m-1}(x_k))]-\tfrac{(b-a)^2}{12n^2}\cdot[\tfrac{\partial H}{\partial x}(x_k,\widetilde{\varphi}_{m-1}(x_k))+\\&+\tfrac{\partial H}{\partial y}(x_k,\widetilde{\varphi}_{m-1}(x_k))(H(x_k,\widetilde{\varphi}_{m-2}(x_k))+f'(x_k))-\tfrac{\partial H}{\partial x}(a,f(a))-\tfrac{\partial H}{\partial y}(a,f(a))\cdot\\&\cdot(H(a,f(a))+f'(a))]+f(x_k)+\widetilde{R}_{m,k}=\widetilde{\varphi}_m(x_k)+\widetilde{R}_{m,k},\forall k=\overline{1,n}.\end{aligned}\tag{6}$$

where we used $\varphi_m'(x)=H(x,\varphi_{m-1}(x))+f\ '(x)$, $\forall x\in[a,b],\forall m\in\mathbb{N}$.

In order to estimate the remainders, if we denote $\overline{M}=M_2(1+M+D_1)$, it we obtain, $|R_{1,k}|\le\tfrac{(b-a)^4M'''}{160n^3}$ and $|\widetilde{R}_{2,k}|\le|R_{2,k}|+[L(b-a)+\tfrac{(b-a)^2\overline{M}}{12n^2}]\cdot|R_{1,k}|\le\tfrac{(b-a)^4M'''}{160n^3}[1+L(b-a)+\tfrac{(b-a)^2\overline{M}}{12n^2}],\forall k=\overline{1,n}$.

t_k	Third order method ε =0.0001	Trapezoidal method ε =0.0001
0	$\widetilde{\varphi}_4(0.000000)$=0.000000	$\widetilde{\varphi}_6(0.000000)$=0.000000
1	$\widetilde{\varphi}_4(0.013333)$=0.000181	$\widetilde{\varphi}_6(0.013333)$=0.000181
2	$\widetilde{\varphi}_4(0.026667)$=0.000737	$\widetilde{\varphi}_6(0.026667)$=0.000737
3	$\widetilde{\varphi}_4(0.040000)$=0.001688	$\widetilde{\varphi}_6(0.040000)$=0.001688
4	$\widetilde{\varphi}_4(0.053333)$=0.003055	$\widetilde{\varphi}_6(0.053333)$=0.003055
5	$\widetilde{\varphi}_4(0.066667)$=0.004860	$\widetilde{\varphi}_6(0.066667)$=0.004860
6	$\widetilde{\varphi}_4(0.080000)$=0.007126	$\widetilde{\varphi}_6(0.080000)$=0.007126
7	$\widetilde{\varphi}_4(0.093333)$=0.009875	$\widetilde{\varphi}_6(0.093333)$=0.009875
8	$\widetilde{\varphi}_4(0.106667)$=0.013134	$\widetilde{\varphi}_6(0.106667)$=0.013134
9	$\widetilde{\varphi}_4(0.120000)$=0.016927	$\widetilde{\varphi}_6(0.120000)$=0.016927
10	$\widetilde{\varphi}_4(0.133333)$=0.021281	$\widetilde{\varphi}_6(0.133333)$=0.021281
11	$\widetilde{\varphi}_4(0.146667)$=0.026224	$\widetilde{\varphi}_6(0.146667)$=0.026224
12	$\widetilde{\varphi}_4(0.160000)$=0.031784	$\widetilde{\varphi}_6(0.160000)$=0.031784
13	$\widetilde{\varphi}_4(0.173333)$=0.037992	$\widetilde{\varphi}_6(0.173333)$=0.037992
14	$\widetilde{\varphi}_4(0.186667)$=0.044878	$\widetilde{\varphi}_6(0.186667)$=0.044879
15	$\widetilde{\varphi}_4(0.200000)$=0.052476	$\widetilde{\varphi}_6(0.200000)$=0.052476

TABLE 1

For $\forall m \in \mathbb{N}, m \geq 3$ and $\forall k = \overline{1,n}$, we obtain by induction,

$$|\widetilde{R}_{m,k}| \leq \frac{(b-a)^4 M^{'''} \cdot (1 - [L(b-a) + \frac{(b-a)^2\overline{M}}{12n^2}]^m)}{160n^3 \cdot (1 - [L(b-a) + \frac{(b-a)^2\overline{M}}{12n^2}])}.$$

So, the remainder estimation is

$$|\widetilde{R}_{m,k}| \leq \frac{(b-a)^4 M^{'''}}{160n^3(1 - [L(b-a) + \frac{(b-a)^2\overline{M}}{12n^2}])}, \ \forall m \in \mathbb{N}, m \geq 3, \ \forall k = \overline{1,n}. \tag{7}$$

In this way we obtain the main result of our paper

Theorem 2. *Under the hypotheses of Theorem 1, if* $L(b-a) + \frac{(b-a)^2\overline{M}}{12n^2} < 1$, $H \in C^3([a,b] \times [-r,r])$ *and* $f \in C^3[a,b]$ *the exact solution* φ *of equation (1) is approximated by the sequence* $(\widetilde{\varphi}_m(x_k))_{m \in \mathbb{N}^*}$, *given by (6) on the knots* x_k, $k = \overline{1,n}$, *with the error estimation*

$$|\varphi(x_k) - \widetilde{\varphi}_m(x_k)| \leq \frac{R \cdot e^{10L(b-a)}}{9 \cdot 10^{m-1}} + \frac{(b-a)^4 M^{'''}}{160n^3(1 - [L(b-a) + \frac{(b-a)^2\overline{M}}{12n^2}])} \tag{8}$$

$,\forall k = \overline{1,n}, \forall m \in \mathbb{N}^*, m \geq 2.$

Proof. We have

$$|\varphi(x_k) - \widetilde{\varphi}_m(x_k)| \leq |\varphi(x_k) - \varphi_m(x_k)| + |\varphi_m(x_k) - \widetilde{\varphi}_m(x_k)|.$$

Since, $|\varphi_m(x_k) - \widetilde{\varphi}_m(x_k)| = |\widetilde{R}_{m,k}|$, $\forall k = \overline{1,n}, \forall m \in \mathbb{N}^*$, from (7) and (3) we obtain (8). □

As numerical example we choose $a = 0, b = 0.5$, $f(x) = x^2$, $H(x,y) = x^2 + 3\sin y$, $n = 15$,and obtain, in the following table, the values $\widetilde{\varphi}_m(x_k)$, $k = \overline{1,n}$,where $m \in \mathbb{N}^*$ is such that $|\widetilde{\varphi}_m(x_k) - \widetilde{\varphi}_{m-1}(x_k)| < \varepsilon$ for each $k = \overline{1,n}$. Comparing with this third order method, we need only 4 iterations while the trapezoidal method requires 6 iterations.

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Existence result and numerical method for a delay integro-differential equation arising in infectious diseases

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Abstract. In this paper we consider a delay integro-differential equation which is a model for the spread of certain infectious disease. We obtain an existence and uniqueness theorem of the positive solution for the considered equation. Afterward we give a numerical method to approximate this solution.

Keywords: delay integro-differential equation, Perov's fixed point theorem, successive approximations.

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1. Introduction

In this paper we consider a model for the spread of certain infections disease with a contact rate that varies seasonally. This model is governed by the following integro-differential equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s), x'(s))\, ds \tag{1}$$

where:

(i) $0 \leq t \leq T$, with $T = l\tau$ for a positive integer l;
(ii) $x(t)$ is the proportion of infectives in the population at time t;
(iii) τ is the length of time in which an individual remainds infectious;
(iv) $x'(t)$ is the speed of infection spreading at moment t;
(v) $f(t, x(t), x'(t))$ is the proportion of new infectives per unit time.

We study the existence and uniqueness of a positive solution for equation (1) and give a numerical method to approximate this solution.

A similary integral equation which models the same problem

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds \tag{2}$$

has been considered in [3], [5], [12], [2], [15], [12] where the sufficent conditions are given for the existence of nontrivial periodic nonnegative and continous solutions for this equation in case of a periodic contact rate: $f(t+\omega, x) = f(t, x)$, $f(t, 0) = 0$. The tools were: Banach fixed point principle in [12], topological fixed point theorems in [3], [5], [2], [15], fixed point index theory in [5] and monotone tehnique in [5], [2], [12]. Also, a system of integral equations in the form (2) has been studied in [2] and [13] using: the monotone tehnique in [2] and Perov fixed point theorem and data dependence on the parameter in [13]. The existence of a positive solution of (2) has been studied in [1] using the Leray-Schauder continuation principle. Using the Lipschitz's condition, in [6], the existence and uniqueness of the positive, bounded solution for (2) is obtained and a numerical method to approximate this solution is given. This numerical method is based on the sequence of successive approximations and on the trapezoidal quadrature rule.

In the following, if X is a nonempty set, by a generalized metric d on X we understand a function $d : X \times X \to \mathbb{R}^n$ which fulfils the following:

(gm1) $0_{\mathbb{R}^n} \leq d(x, y), \forall x, y \in X$ and $d(x, y) = 0_{\mathbb{R}^n} \Leftrightarrow x = y$;
(gm2) $d(x, y) = d(y, x), \forall x, y \in X$;

(gm3) $d(x,y) \leq d(x,z) + d(z,y), \forall x,y,z \in X,$

where, for $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ from $\mathbb{R}^n$, we have $x \leq y \Leftrightarrow x_i \leq y_i$, for any $i = \overline{1,n}$. The pair (X,d) will be called generalized metric space.

2. Existence and uniqueness

Denote by X the product space $C^1[-\tau,T] \times C[-\tau,T]$ which is generalized metric space with the Cauchy metric type given by

$$d_C : X \times X \to \mathbb{R}^2,\ d_C((x_1,y_1),(x_2,y_2)) = (\|x_1 - x_2\|, \|y_1 - y_2\|),$$

where $\|u\| = \max_{t\in[-\tau,T]} |u(t)|$, for any $u \in C^1[-\tau,T] \cap C[-\tau,T]$.

Also, we consider $X_+ = \{(x,y) \in X:\ x(t) \geq 0, \text{ for any } t \in [-\tau,T]\}$. Since X_+ is closed in X and X is a complete metric space we infer that X_+ is a complete metric space too.

In the following we study the initial value problem

$$x(t) = \begin{cases} \int_{t-\tau}^{t} f(s, x(s), x'(s))\, ds, \text{ for } t \in [0,T] \\ \varphi(t), \text{ for } t \in [-\tau, 0] \end{cases} \tag{3}$$

for known $\varphi \in C^1[-\tau,0]$. We suppose that the following conditions hold:

- (C_1) (boundness conditions) $\exists m, M \in \mathbb{R}$ such that $0 \leq m \leq f(t,u,v) \leq M, \forall t \in [-\tau,T], \forall u \in [0,\infty), \forall v \in \mathbb{R}$, $\exists m_1, M_1 \in \mathbb{R}$ such that $0 \leq m_1 \leq \varphi(t) \leq M_1, \forall t \in [-\tau,0]$;
- (C_2) (Lipschitz condition) $f \in C([-\tau,T] \times [0,\infty) \times \mathbb{R})$ and $\exists \alpha, \beta > 0$ such that $|f(t,u,v) - f(t,u',v')| \leq \alpha |u - u'| + \beta |v - v'|, \forall t \in [-\tau,T], \forall u,u' \in [0,\infty), \forall v,v' \in \mathbb{R}$;
- (C_3) (compatibility conditions) $\varphi(0) = \int_{-\tau}^{0} f(s, \varphi(s), \varphi'(s))\, ds$ and $\varphi'(0) = f(0, \varphi(0), \varphi'(0)) - f(-\tau, \varphi(-\tau), \varphi'(-\tau))$.

We study the existence and uniqueness of the positive solution of (18) in the above hypotheses by using Perov's fixed point theorem [8] (see also [4], [7])

Theorem 1. *(PEROV) Let (X,d) be a complete generalized metric space. If $T : X \to X$ is a map for which there exists a matrix $A \in \mathcal{M}_2(\mathbb{R})$ such that:*

(i) $d(T(x), T(y)) \leq Ad(x,y), \forall x,y \in X$;
(ii) *the eigenvalues of A lie in the unit disc from $\mathbb{R}^2$,*

then:

(1) *T has a unique fixed point x^*;*
(2) *the sequence of successive approximations $x_m = T^m(x_0)$ converges to x^* for any $x_0 \in X$;*
(3) *the estimation*

$$d(x_m, x^*) \leq A^m (I_2 - A)^{-1} d(x_0, x_1), \forall m \in \mathbb{N}^* \text{ holds.}$$

If we differentiate (18) with respect t and denote $y(t) = x'(t)$ we obtain

$$y(t) = f(t, x(t), y(t)) - f(t - \tau, x(t-\tau), y(t-\tau)),$$

for $t \in [0,T]$ and $y(t) = \varphi'(t)$, for $t \in [-\tau, 0]$.

Let $T : X_+ \to X$ be the map given by

$$T(x,y)(t) = \begin{cases} \begin{pmatrix} \int_{t-\tau}^{t} f(s, x(s), y(s))\, ds, \\ f(t, x(t), y(t)) - f(t-\tau, x(t-\tau), y(t-\tau)) \end{pmatrix}, t \in [0,T] \\ (\varphi(t), \varphi'(t)),\ t \in [-\tau, 0]. \end{cases} \tag{4}$$

From hypotheses (C_1) we have that $T(X_+) \subseteq X_+$. For any $(x_1,y_1),(x_2,y_2) \in X_+$ we estimate $d_C((x_1,y_1),(x_2,y_2))$.

If $t \in [0,T]$ we have

$d_C\left(T\left(x_1,y_1\right),T\left(x_2,y_2\right)\right) =$

$$\left(\max_{t\in[0,T]}\left|\int_{t-\tau}^{t} f\left(s,x_1\left(s\right),y_1\left(s\right)\right)ds - \int_{t-\tau}^{t} f\left(s,x_2\left(s\right),y_2\left(s\right)\right)ds\right|,\right.$$
$$\left.\max_{t\in[0,T]} \left| f\left(t,x_1\left(t\right),y_1\left(t\right)\right) - f\left(t-\tau,x_1\left(t-\tau\right),y_1\left(t-\tau\right)\right) - f\left(t,x_2\left(t\right),y_2\left(t\right)\right) + f\left(t-\tau,x_2\left(t-\tau\right),y_2\left(t-\tau\right)\right)\right|\right).$$

$$\left|\int_{t-\tau}^{t} f\left(s,x_1\left(s\right),y_1\left(s\right)\right)ds - \int_{t-\tau}^{t} f\left(s,x_2\left(s\right),y_2\left(s\right)\right)ds\right| \overset{C_2}{\leq}$$

$$\leq \int_{t-\tau}^{t} \left(\alpha\left|x_1\left(s\right)-x_2\left(s\right)\right| + \beta\left|y_1\left(s\right)-y_2\left(s\right)\right|\right)ds \leq$$

$$\leq \alpha\tau\left\|x_1-x_2\right\| + \beta\tau\left\|y_1-y_2\right\|.$$

Hence

$$\max_{t\in[0,T]}\left|\int_{t-\tau}^{t} f\left(s,x_1\left(s\right),y_1\left(s\right)\right)ds - \int_{t-\tau}^{t} f\left(s,x_2\left(s\right),y_2\left(s\right)\right)ds\right| \leq \tag{5}$$
$$\leq \alpha\tau\left\|x_1-x_2\right\| + \beta\tau\left\|y_1-y_2\right\|.$$

$$\left|\left(f\left(t,x_1\left(t\right),y_1\left(t\right)\right) - f\left(t-\tau,x_1\left(t-\tau\right),y_1\left(t-\tau\right)\right)\right) -\right.$$
$$\left. - \left(f\left(t,x_2\left(t\right),y_2\left(t\right)\right) - f\left(t-\tau,x_2\left(t-\tau\right),y_2\left(t-\tau\right)\right)\right)\right| \overset{C_2}{\leq}$$
$$\leq \alpha\left|x_1\left(t\right)-x_2\left(t\right)\right| + \beta\left|y_1\left(t\right)-y_2\left(t\right)\right| +$$
$$+\alpha\left|x_1\left(t-\tau\right)-x_2\left(t-\tau\right)\right| + \beta\left|y_1\left(t-\tau\right)-y_2\left(t-\tau\right)\right| \leq$$
$$\leq 2\alpha\left\|x_1-x_2\right\| + 2\beta\left\|y_1-y_2\right\|.$$

Hence

$$\max_{t\in[0,T]}\left|f\left(t,x_1\left(t\right),y_1\left(t\right)\right) - f\left(t-\tau,x_1\left(t-\tau\right),y_1\left(t-\tau\right)\right) -\right. \tag{6}$$
$$\left. -f\left(t,x_2\left(t\right),y_2\left(t\right)\right) + f\left(t-\tau,x_2\left(t-\tau\right),y_2\left(t-\tau\right)\right)\right| e^{-\theta\left(t+\tau\right)} \leq$$
$$\leq 2\alpha\left\|x_1-x_2\right\| + 2\beta\left\|y_1-y_2\right\|.$$

If $t \in [-\tau,0]$ we have

$$d_C\left(T\left(x_1,y_1\right),T\left(x_2,y_2\right)\right)\left(t\right) = \left(0,0\right). \tag{7}$$

From (5), (6), (7) we have

$$d_C\left(T\left(x_1,y_1\right),T\left(x_2,y_2\right)\right) \leq A d_C\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right), \tag{8}$$

where $A = \begin{pmatrix} \alpha\tau & \beta\tau \\ 2\alpha & 2\beta \end{pmatrix}$. The eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 2\beta + \alpha\tau$.

Here is our main result.

Theorem 2. *If the conditions C_1, C_2, C_3 for (18) hold and $2\beta + \alpha\tau < 1$ in C_2 then the initial value problem (18) has an unique bounded solution x^*, in X_+.*

Proof. By Theorem 1 for the map T defined by (19), from (8) and by Theorem 1 the existence and uniqueness of solution follow. Denote by $x^*\left(t\right) = \left(x_1^*\left(t\right), y_1^*\left(t\right)\right)$ this solution.

From C_1 we have $0 < m\tau \leq x_1^*\left(t\right) = \int_{t-\tau}^{t} f\left(s,x_1^*\left(s\right),y_1^*\left(s\right)\right)ds \leq M\tau$, $\forall t \in [0,T]$ and $0 < m_1 \leq x_1^*\left(t\right) \leq M_1$, $\forall t \in [-\tau,0]$, hence x_1^* is bounded.

Let us prove that $\left(x_1^*\right)'\left(t\right) = y_1^*\left(t\right), \forall t \in [-\tau,T]$. To this aim, in the case $t \in [-\tau,0]$, from (19) we have, $T\left(x_1^*\left(t\right),y_1^*\left(t\right)\right) = \left(\varphi(t),\varphi'(t)\right) = \left(x_1^*\left(t\right),y_1^*\left(t\right)\right)$, therefor, $x_1^*\left(t\right) = \varphi(t)$ and $y_1^*\left(t\right) = \varphi'(t)$, which

means $(x_1^*)'(t) = y_1^*(t)$. In the case $t \in [0,T]$, from (19) we have, $(x_1^*(t), y_1^*(t)) = T(x_1^*(t), y_1^*(t))$ and

$$T(x_1^*(t), y_1^*(t)) = \begin{pmatrix} \int_{t-\tau}^{t} f(s, x(s), y(s))\, ds, \\ f(t, x(t), y(t)) - f(t-\tau, x(t-\tau), y(t-\tau)) \end{pmatrix}$$

and then

$$\begin{cases} x_1^*(t) = \int_{t-\tau}^{t} f(s, x(s), y(s))\, ds \\ y_1^*(t) = f(t, x(t), y(t)) - f(t-\tau, x(t-\tau), y(t-\tau)) \end{cases},$$

which means $(x_1^*)'(t) = y_1^*(t)$. □

Corollary 3. *Under the hypotheses of Theorem 6 the solution $x^*(t)$ of (18), which is obtained by the successive approximations method starting with an arbitrary point $x^0 = (x_0, y_0) \in X_+$, satisfies the inequality*

$$d_C(x^m, x^*) \leq \frac{\lambda_2^{m-1}}{1-\lambda_2} \begin{pmatrix} \alpha\tau & \beta\tau \\ 2\alpha & 2\beta \end{pmatrix} d_C(x^1, x^0). \tag{9}$$

where $x^1 = T(x^0)$, $x^m = T(x^{m-1})$, $x^m = (x_m, y_m), \forall m \in \mathbb{N}^*$, $x^0 = (x_0, y_0)$.

Proof. By Theorem 1, under the hypotheses of Theorem6 we have that $d_B(x^m, x^*) \leq A^m (I-A)^{-1} d_B(x^1, x^0)$, $\forall m \in \mathbb{N}^*$. For the matrix A, given in (8), we have $A^m = \lambda_2^{m-1} A$, $\forall m \in \mathbb{N}^*$ and
$(I-A)^{-1} = \frac{1}{1-\lambda_2} \begin{pmatrix} 1-2\beta & \beta\tau \\ 2\alpha & 1-\alpha\tau \end{pmatrix}$. □

3. The approximation of the solution

By Theorem 6 the solution of (18) can be obtained by successive approximation method starting with an arbitrary element of X_+. For this reason, we compute the terms by means of the sequence of successive approximations $x^m(t) = (x_m(t), y_m(t))$, $\forall m \in \mathbb{N}$. In the case $t \in [-\tau, 0]$ we have that $x_m(t) = \varphi(t)$ and $y_m(t) = \varphi'(t)$, for $\forall m \in \mathbb{N}$. In the case $t \in [0,T]$ we have that $x_0(t) = \varphi(0)$, $y_0(t) = \varphi'(0)$,

$$x_m(t) = \int_{t-\tau}^{t} f(s, x_{m-1}(s), y_{m-1}(s))\, ds, \ \forall m \in \mathbb{N}^*, \tag{10}$$

$$y_m(t) = f(t, x_{m-1}(t), y_{m-1}(t)) - f(t-\tau, x_{m-1}(t-\tau), y_{m-1}(t-\tau)), \ \forall m \in \mathbb{N}^*.$$

In order to compute the above integrals we use a quadrature rule. Let Δ_n be a uniform partition of the interval $[-\tau, 0]$, $\Delta_n : -\tau = t_0 < t_1 < ... < t_n = 0$, with $h = t_i - t_{i-1}$, $\forall i = \overline{1,n}$.

Suppose that the values of the function φ on the knots t_i, $i = \overline{0,n}$, and the values of the function φ' on the same knots are known.

In order to compute the integrals from (10) we use the trapezoidal type quadrature rule (as in [1]):

$$\int_a^b F(t)\, dt = \frac{b-a}{2n} \left[F(a) + 2\sum_{i=1}^{n-1} F(t_i) + F(b) \right] + R_n(F), \tag{11}$$

where $t_i = a + \frac{(b-a)}{n} i$, $i = \overline{0,n}$ and $|R_n(F)| \leq \frac{(b-a)}{2} \omega(F, \frac{b-a}{2n})$, with the modulus of continuity of F. Since $\lim_{n\to\infty} \frac{b-a}{2n} = 0$ and F is continuous, we have that $\lim_{n\to\infty} \omega(F, \frac{b-a}{2n}) = 0$.

Let $F : [-\tau, T] \to \mathbb{R}$, $F(t) = f(t, x(t), y(t))$ which is continous by virtue of condition C_2 and, for any $m \in \mathbb{N}$, denote $F_m(t) = f(t, x_m(t), y_m(t))$, for $t \in [-\tau, T]$.

For the interval $[0,T]$ where $T = l\tau$, consider the uniform partition $\Delta_n' : 0 = t_n < t_{n+1} < ... < t_q = T$, with $q = n + 1 + l \cdot n$ and $t_{j+1} - t_j = h = \frac{\tau}{n}$, $\forall j = \overline{n, q-1}$. Obviously $t_k - \tau = t_{k-n}$, $k = \overline{n,q}$ and

$\Delta_n \cup \Delta_n'$ is a uniform partition of the interval $[-\tau, T]$. For any $k = \overline{n+1,q}$ in the interval $[t_k - \tau, t_k]$ we find n knots of the partition $\Delta_n \cup \Delta_n'$. Applying (11) to compute the integrals (10) we obtain

$$\begin{aligned} & x_m(t_k) = \int_{t_k-\tau}^{t_k} f(s, x_{m-1}(s), y_{m-1}(s))\, ds = \\ & = \tfrac{\tau}{2n}[F_{m-1}(t_k) + 2\sum_{j=1}^{n-1} F_{m-1}(t_{k+j} - \tau) + F_{m-1}(t_k - \tau)] + r_{m,k}(f), \end{aligned} \tag{12}$$

$\forall k = \overline{n+1,q}$, $\forall m \in \mathbb{N}$, with the estimation $|r_{m,k}(f)| \leq \frac{\tau}{2}\omega(F_{m-1}, \frac{\tau}{2n})$ which does not depend on k .

On the knots we have the values

$$x_0(t_k) = \begin{cases} \varphi(t_k),\ k = \overline{0,n} \\ \varphi(0),\ k = \overline{n+1,q} \end{cases}, \quad y_0(t_k) = \begin{cases} \varphi'(t_k),\ k = \overline{0,n} \\ \varphi'(0),\ k = \overline{n+1,q} \end{cases},$$

and $y_m(t_k) = F_{m-1}(t_k) - F_{m-1}(t_k - \tau)$, $\forall k = \overline{n+1,q}$, $\forall m \in \mathbb{N}^*$.

Using (12) for $m = 1, 2$ we have

$x_1(t_k) = \frac{\tau}{2n}[F_0(t_k) + 2\sum_{j=1}^{n-1} F_0(t_{k+j} - \tau) + F_0(t_k - \tau)] + r_{1,k}(f) \equiv \overline{x_1}(t_k) + r_{1,k}(f)$, $\forall k = \overline{n+1,q}$.

We have $y_1(t_k) = F_0(t_k) - F_0(t_k - \tau) \equiv \overline{y_1}(t_k)$, $\forall k = \overline{n+1,q}$, and $x_2(t_k) = \overline{x_2(t_k)} + \overline{r_{2,k}(f)}, \forall k = \overline{n+1,q}$.

$y_2(t_k) = f(t_k, \overline{x_1(t_k)}, y_1(t_k)) - f(t_k - \tau, \overline{x_1(t_k - \tau)}, y_1(t_k - \tau)) + \delta_{2,k}(f) = \overline{y_2(t_k)} + \delta_{2,k}(f)$. For the remainders we have the estimates

$$\begin{aligned} & \left|\overline{r_{2,k}(f)}\right| \leq \tfrac{\tau}{2n}[\alpha |r_{1,k}(f)| + \alpha |r_{1,k-n}(f)| + 2\sum_{j=1}^{n-1} \alpha |r_{1,k+j-n}(f)|] + \\ & + |r_{2,k}(f)| \leq \tfrac{\alpha}{2}\tau^2\omega(F_0, \tfrac{\tau}{2n}) + \tfrac{\tau}{2}\omega(F_1, \tfrac{\tau}{2n}), \forall k = \overline{n+1,q}, \end{aligned} \tag{13}$$

and $|\delta_{2,k}(f)| \leq \alpha |r_{1,k}(f)| + \alpha |r_{1,k-n}(f)| \leq \alpha\tau \cdot \omega(F_0, \frac{\tau}{2n})$. By induction, for $m \geq 3$ we have

$$\begin{aligned} & x_m(t_k) = \tfrac{\tau}{2n}[F_{m-1}(t_k) + 2\sum_{j=1}^{n-1} F_{m-1}(t_{k+j} - \tau) + F_{m-1}(t_k - \tau)] + r_{m,k}(f) = \\ & \tfrac{\tau}{2n}[f(t_k, \overline{x_{m-1}(t_k)} + \overline{r_{m-1,k}}(f), \overline{y_{m-1}(t_k)} + \delta_{m-1,k}(f)) + f(t_{k-n}, \overline{x_{m-1}(t_{k-n})} \\ & + \overline{r_{m-1,k-n}}(f), \overline{y_{m-1}(t_{k-n})} + \delta_{m-1,k-n}(f)) + 2\sum_{j=1}^{n-1} f(t_{k+j-n}, \overline{x_{m-1}(t_{k+j-n})} \\ & + \overline{r_{m-1,k+j-n}}(f), \overline{y_{m-1}(t_{k+j-n})} + \delta_{m-1,k+j-n}(f))] + r_{m,k}(f) = \tfrac{\tau}{2n}[f(t_k, \\ & , \overline{x_{m-1}(t_k)}, , y_{m-1}(t_k)) + f(t_{k-n}, \overline{x_{m-1}(t_{k-n})}, \overline{y_{m-1}(t_{k-n})}) + 2\sum_{j=1}^{n-1} f(t_{k+j-n}, \\ & , \overline{x_{m-1}(t_{k+j-n})}, \overline{y_{m-1}(t_{k+j-n})})] + \overline{r_{m,k}(f)} = \overline{x_m(t_k)} + \overline{r_{m,k}(f)}, \end{aligned} \tag{14}$$

and $y_m(t_k) = f(t_k, \overline{x_{m-1}(t_k)}, \overline{y_{m-1}(t_k)}) - f(t_k - \tau, \overline{x_{m-1}(t_k - \tau)}, \overline{y_{m-1}(t_k - \tau)}) + \delta_{m,k}(f) = \overline{y_m(t_k)} + \delta_{m,k}(f)$, $\forall k = \overline{n+1,q}$. To estimate the remainders, for $m \in \mathbb{N}, m \geq 3$ and for any $k = \overline{n+1,q}$, we have the reccurent relations:

$|\delta_{m,k}(f)| \leq 2\alpha \left|\overline{r_{m-1,k}(f)}\right| + 2\beta |\delta_{m-1,k}(f)|$

$\left|\overline{r_{m,k}(f)}\right| \leq \tau\alpha \left|\overline{r_{m-1,k}(f)}\right| + \tau\beta |\delta_{m-1,k}(f)| + |r_{m,k}(f)|$.

Denote by ρ_m the the least upper bound for $\left|\overline{r_{m,k}(f)}\right|$ and $|\delta_{m,k}(f)|$, $\forall m \in \mathbb{N}^*, \forall k = \overline{n+1,q}$.

We impose the following condition: $\exists \gamma \geq 0$ such that

$$|f(t_1, u, v) - f(t_2, u, v)| \leq \gamma |t_1 - t_2|, \ \forall t_1, t_2 \in [-\tau, T], \forall u \geq 0, \forall v \in \mathbb{R}. \tag{15}$$

If $\varphi \in C^2[-\tau, 0]$, let us $D, D' \in \mathbb{R}$, $D = \|\varphi'\|, D' = \|\varphi''\|$. Then in the above condition, the functions F_0 and F_1 are Lipschitzian with Lipschitz's constants $L_0 = \gamma + \alpha \cdot D + \beta \cdot D'$, $L_1 = \max(L_0, \gamma + \beta + \beta L_0 + 2\alpha M, \gamma + 2\alpha M + 2\beta\gamma)$. By induction for $m \in \mathbb{N}^*$, we suppose that F_{m-1} is Lipschitzian with a constant L_{m-1}, and obtain successively, $|x_m(t_1) - x_m(t_2)| \leq 2M \cdot |t_1 - t_2|, \forall t_1, t_2 \in [-\tau, T]$,

$$\begin{aligned} & |y_m(t_1) - y_m(t_2)| \leq |F_{m-1}(t_1) - F_{m-1}(t_2)| + \\ & + |F_{m-1}(t_1 - \tau) - F_{m-1}(t_2 - \tau)| \leq 2L_{m-1} |t_1 - t_2|, \ \forall t_1, t_2 \in [0, T], \end{aligned}$$

and since $\varphi \in C^2[-\tau,0]$ and $\|\varphi''\| \leq L_{m-1}$we obtain that $|y_m(t_1)-y_m(t_2)| \leq 2L_{m-1}|t_1-t_2|$, $\forall t_1,t_2 \in [-\tau,T]$. Then for $\forall t_1,t_2 \in [-\tau,T]$ we have:

$$|F_m(t_1)-F_m(t_2)| \leq \gamma|t_1-t_2|+\alpha|x_m(t_1)-x_m(t_2)|+$$
$$+\beta|y_m(t_1)-y_m(t_2)| \leq (\gamma+2\alpha M+2\beta L_{m-1})\cdot|t_1-t_2|.$$

Now, we can infer that F_m is Lipschitzian with a Lipschitz constant $L_m=\gamma+2\alpha M+2\beta L_{m-1}$, $\forall m \in \mathbb{N}^*$. We can see that $L_{m+1}-L_m=2\beta(L_m-L_{m-1})=...=(2\beta)^m(L_1-L_0)\geq 0$, since $L_1 \geq L_0$, $\forall m \in \mathbb{N}^*$, and then the sequence $(L_m)_{m\in\mathbb{N}^*}$ is increasing. On the other hand we have,

$$|L_{m+p}-L_m| \leq |L_{m+p}-L_{m+p-1}|+...+|L_{m+1}-L_m| \leq [(2\beta)^m+$$
$$+(2\beta)^{m+1}+...+(2\beta)^{m+p-1}]\cdot|L_1-L_0|=(2\beta)^m[1+2\beta+$$
$$+...+(2\beta)^{p-1}]\cdot|L_1-L_0|=(2\beta)^m\cdot\tfrac{1-(2\beta)^m}{1+2\beta}\cdot|L_1-L_0|.$$

If $\beta<\frac{1}{2}$ then $\lim\limits_{p\to\infty}|L_{m+p}-L_m|=0$ and the sequence $(L_m)_{m\in\mathbb{N}^*}$ converge. Then $\exists L\geq 0$ such that $L=\lim\limits_{m\to\infty}L_m$. Consequently, $L_m\leq L$, $\forall m\in\mathbb{N}$ and the estimation $|r_{m,k}(f)|\leq\frac{\tau}{2}\omega(F_{m-1},\frac{\tau}{2n})$ lead to $|r_{m,k}(f)|\leq\frac{L\tau^2}{4n}$, $\forall m\in\mathbb{N}^*,\forall k=\overline{n+1,q}$. Moreover, if $\alpha<\frac{1}{2}$ and $\tau<1$ then

$$\left|\overline{r_{2,k}(f)}\right|\leq\frac{\alpha}{2}\tau^2\omega(F_0,\frac{\tau}{2n})+\frac{\tau}{2}\omega(F_1,\frac{\tau}{2n})\leq\frac{\tau}{2}(\alpha\tau+1)\frac{L\tau^2}{4n}<(\tau+1)\frac{L\tau^2}{4n}$$

$$|\delta_{2,k}(f)|\leq\alpha|r_{1,k}(f)|+\alpha|r_{1,k-n}(f)|\leq 2\alpha\cdot\frac{\tau}{2}\omega(F_0,\frac{\tau}{2n})\leq\frac{L\tau^2}{4n},$$

and

$$\left|\overline{r_{3,k}(f)}\right|\leq\tau(\alpha+\beta)\cdot\rho_2+\frac{L\tau^2}{4n}<(\tau^2+\tau+1)\frac{L\tau^2}{4n}.$$

By induction, for $m\in\mathbb{N},m\geq 3$ and $\forall k=\overline{n+1,q}$, we have,

$$\begin{aligned}&\left|\overline{r_{m,k}(f)}\right|\leq\tau(a\left|\overline{r_{m-1,k}(f)}\right|+\beta|\delta_{m-1,k}(f)|)+|r_{m,k}(f)|\leq\\&\leq\tau(\alpha+\beta)\rho_{m-1}+\tfrac{L\tau^2}{4n}\leq\tau(\tau^{m-2}+...+\tau^2+\tau+1)\tfrac{L\tau^2}{4n}+\tfrac{L\tau^2}{4n}=\\&=\left(\tfrac{1-\tau^m}{1-\tau}\right)\cdot\tfrac{L\tau^2}{4n}\leq\tfrac{L\tau^2}{4n(1-\tau)}\ .\end{aligned}\tag{16}$$

Theorem 4. *Under the hypotheses of Theorem6 and condition 15, if* $\max(\alpha,\beta)<\frac{1}{2},\tau<1$ *and* $\varphi\in C^2[-\tau,0]$, *then the solution* x_1^* *of the initial value problem (18) is approximated by the terms of the sequence* $\left(\overline{x_m(t_k)}\right)_{m\in N}$ *from (14) with the error*

$$\left|x_1^*(t_k)-\overline{x_m(t_k)}\right|\leq\frac{\lambda_2^{m-1}}{1-\lambda_2}\cdot[(1-2\beta)(M\tau+M_1)+\beta\tau(2M+D)]+\frac{L\tau^2}{4n(1-\tau)}$$

$,\forall m\in\mathbb{N}^*,\forall k=\overline{n+1,q}$.

Proof. It follows from Corollary 3 and inequality (16). □

3.1. **Example.** Consider the functions $f:[-\tau,T]\times[0,\infty)\times\mathbb{R}\to\mathbb{R}$, $\varphi:[-\tau,0]\to\mathbb{R}$, $f(t,y,z)=at+b+\frac{dz^2+cy}{z^2+y^2+K^2\zeta^2}$, $\varphi(t)=K(t+\tau)$, with $a,b,c,d,K,\zeta>0$. Such functions satisfy the conditions C_1-C_3 and (15). We illustrate the algorithm, choosing $\tau=\frac{3}{4},T=3,K=10$, $c=\frac{1}{10}$, $d=\frac{1}{6},\zeta=\sqrt[2]{3},a=\frac{10}{3}+\frac{dK\tau-c(\zeta^2+1)}{(\zeta^2+1)(\tau^2+\zeta^2+1)}$, $b=10+\frac{3}{2}a$ and $n=6$. Then $q=31$ and we obtain the values $\overline{x_m(t_k)}$ and $\overline{y_m(t_k)}$ from (14), where $m\in\mathbb{N}$ is such that for each $k=\overline{n+1,q}$ we have for an $\varepsilon>0$:

$$\left|\overline{x_m(t_k)}-\overline{x_{m-1}(t_k)}\right|<\varepsilon\text{ and }\left|\overline{y_m(t_k)}-\overline{y_{m-1}(t_k)}\right|<\varepsilon,\ \forall k=\overline{n+1,q}.$$

1	1.25	10.0
2	2.5	10.0
3	3.75	10.0
4	5.0	10.0
5	6.25	10.0
6	7.5	10.0
7	45.681887	10.001657
8	54.980827	9.999766
9	64.279593	9.99887
10	73.57829	9.998671
11	82.876995	9.998981
12	92.175762	9.999667
13	101.476399	10.028906
14	110.778973	10.030654
15	120.081726	10.031778
16	129.384597	10.032533
17	138.687548	10.03306
18	147.990555	10.033438
19	157.293603	10.033709
20	166.596681	10.033922
21	175.899783	10.034086
22	185.202903	10.034215
23	194.406038	10.034318
24	203.809184	10.034402
25	213.11234	10.03447
26	222.415504	10.034526
27	231.718674	10.034573
28	241.021849	10.034613
29	250.325029	10.034647
30	259.628213	10.034676
31	268.931401	10.034701

TABLE 1

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Isomorphism for transitive groupoid C^*-algebras

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Abstract. We prove that the C^*-algebra of the locally compact second countable transitive groupoid G is $*$-isomorphic to the C^*-algebra of the groupoid $G^{(0)}\times H\times G^{(0)}$ endowed to the Haar system $\{\varepsilon_u\times\nu_H\times\mu\}$, where H is the isotropy group G_u^u at any unit $u\in G^{(0)}$, ε_u is the unit point mass at u, ν_H is a Haar measure on H, and μ is a quasi-invariant Radon measure with respect to the Haar system of G. The C^* -algebra of $G^{(0)}\times H\times G^{(0)}$ is easily seen to be $*$-isomorphic to $C^*(H)\otimes\mathcal{K}\left(L^2(\mu)\right)$, where $C^*(H)$ denotes the group C^*-algebra of H, and $\mathcal{K}\left(L^2(\mu)\right)$ denotes the compact operators on $L^2(\mu)$. Therefore the C^*-algebra of G is $*$-isomorphic to $C^*(H)\otimes\mathcal{K}\left(L^2(\mu)\right)$. Thus we regain a result of P. Muhly, J. Renault and D. Williams (Theorem 3.1, p. 16 [7]).

Keywords: transitive groupoid, trivial groupoid, groupoid isomorphism, $*$-isomorphism, C^*-algebra.

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1. Introduction

The construction of the C^*-algebra of a groupoid extends the case of a group. The space of continuous functions with compact support on groupoid is made into a $*$-algebra and endowed with the smallest C^*-norm making its representations continuous. For this $*$-algebra the multiplication is convolution. For defining the convolution on a locally compact groupoid, one needs an analogue of Haar measure on locally compact groups. This analogue is a system of measures, called Haar system, subject to suitable invariance and smoothness conditions called respectively "left invariance" and "continuity". Unlike the case of locally compact group, Haar system on groupoid need not exist, and if it does, it will not usually be unique. However, on locally compact second countable groupoids one can construct systems of measures satisfying "left invariance" condition. But the continuity assumption has topological consequences for groupoid. It entails that the range map (and hence the domain map) is open (Proposition I. 4 [13]). A. K. Seda ([12]) has proved that the "continuity" condition is crucial in construction of the groupoid C^*-algebra . In this paper we only use locally compact second countable transitive groupoids. For this kind of groupoids the "continuity" condition is a consequence of the "left invariance" condition (Theorem 4.4 [2] or Theorem 2.2 B p. 8 [7] and Theorem 2 p. 430 [11]). Thus for this kind of groupoids always there are Haar systems. As in the general case, the Haar system need not to be unique. A result of P. Muhly, J. Renault and D. Williams (Theorem 2.8 p. 10 [7]) states that the C^*-algebras associated with two different Haar systems on a locally compact second countable groupoid (not necessarily transitive) are strongly Morita equivalent. It is not known if they are $*$-isomorphic. For a transitive groupoid G, P. Muhly, J. Renault and D. Williams have showed that then the C^*-algebra of G is isomorphic to $C^*(H)\otimes\mathcal{K}\left(L^2(\mu)\right)$, where H is the isotropy group G_u^u at any unit $u\in G^{(0)}$, μ is an essentially unique measure on $G^{(0)}$, $C^*(H)$ denotes the group C^*-algebra of H, and $\mathcal{K}\left(L^2(\mu)\right)$ denotes the compact operators on $L^2(\mu)$. For proving that result they firstly established that $C^*(G)$ and $C^*(H)$ are strong Morita equivalent via a $C^*(H)$ module X_1. As a consequence the C^*-algebra of G is the imprimitivity algebra of X_1. Then they needed another $C^*(H)$ module X_2 whose imprimitivity algebra is $C^*(H)\otimes\mathcal{K}\left(L^2(\mu)\right)$ for a suitable measure μ. By this result, it follows that the C^*-algebras associate with two Haar systems on a locally compact second countable transitive groupoid are $*$-isomorphic.

We obtain the isomorphism between the C^*-algebra of G and $C^*(H) \otimes \mathcal{K}\left(L^2(\mu)\right)$ more directly: we show that the C^*-algebra of G is isomorphic with the C^*-algebra of the groupoid $G^{(0)} \times H \times G^{(0)}$ endowed with the Haar system $\{\delta_u \times \nu_H \times \mu\}$,where ν_H is a Haar measure on H, and μ is quasi invariant probability measure with respect to the Haar system of G. In order to prove that, we construct a groupoid Borel isomorphism ϕ between G and $G^{(0)} \times H \times G^{(0)}$ which carry the Haar system of G to a Haar system of $G^{(0)} \times H \times G^{(0)}$ of the form $\{\delta_u \times \nu_H \times \mu\}$.Then we use the fact that any compactly supported Borel bounded function can be viewed as an element of the C^*- algebra (Proposition 4, p. 82, Proposition 5, p. 86 [3]) and we prove that the $f \to \phi \circ f$ can be extended to a $*$-isomorphism of C^*-algebras.

2. Haar systems on G and Haar systems on the trivial groupoid $G^{(0)} \times G_e^e \times G^{(0)}$

Throughout this paper G will stand for a second countable locally compact transitive groupoid and $\left\{\nu^u,\, u \in G^{(0)}\right\}$ a Haar system on G. We use the terminology and notation of [10].

According to Lemma 4.5, p. 277 [9] or Proposition I.3.8 [10] all quasi-invariant measures carried by an orbit $[u]$ are equivalent. Since the groupoid G is transitive, it follows that it has a single orbit, and therefore all quasi-invariant measures are equivalent.

In Section 3 [1], we have proved that we can choose a quasi-invariant probability measure μ such that there exists a family $\left\{\nu_{u,v},\, u, v \in G^{(0)}\right\}$ of σ-finite measures on G with the following properties:

(1) $\nu_{u,v}$ is supported on G_v^u, and $\nu_{u,v} \neq 0$, for all $u, v \in G^{(0)}$.

(2) For all $f \geq 0$ Borel on G,

$$(u,v) \mapsto \int f(y)\, d\nu_{u.v}(y) \quad \left[: G^{(0)} \times G^{(0)} \to \overline{\mathbf{R}}\right]$$

is an extended real-valued Borel function.

(3) For all $f \geq 0$ Borel on G,

$$\int f(xy)\, d\nu_{d(x),v}(y) = \int f(y)\, d\nu_{r(x),v}(y) \quad \text{for all } x \in G,\ v \in G^{(0)}.$$

(4) For all u in $G^{(0)}$, $\nu^u = \int \nu_{u,v} d\mu(v)$.

Moreover, if Δ is the modular function of μ, then Δ may be chosen to be a strict homomorphisms and

5. For all $f \geq 0$ Borel on G,

$$\Delta(x) \int f(yx)\, d\nu_{u,r(x)}(y) = \int f(y)\, d\nu_{u,d(x)}(y) \quad \text{for all } x \in G,\ u \in G^{(0)}.$$

6. For all $f \geq 0$ Borel on G,

$$\int f(y)\, d\nu_{u,v}(y) = \int f\left(y^{-1}\right) \Delta\left(y^{-1}\right) d\nu_{v,u}(y) \text{ for all } u, v \in G^{(0)}.$$

A similar decomposition for more general groupoids can be found in [4].

Let $\delta : G \to \mathbf{R}_+^*$ be a strict homomorphism and $\left\{\beta_{u,v},\, u, v \in G^{(0)}\right\}$a system of measures on G, satisfying:

(i)**:** For all $f \geq 0$ Borel on G,

$$\int f(xy)\, d\beta_{d(x),v}(y) = \int f(y)\, d\beta_{r(x),v}(y)$$

for all $x \in G$, $v \in G^{(0)}$.

(ii)**:** For all $f \geq 0$ Borel on G,

$$\delta(x) \int f(yx)\, d\beta_{u,r(x)}(y) = \int f(y)\, d\beta_{u,d(x)}(y)$$

for all $x \in G$, $u \in G^{(0)}$.

Using the uniqueness of the Haar measure on the locally compact group G_v^v and arguing as in Section 2 [1], it results that there exists a positive function $h : G^{(0)} \to \mathbf{R}_+^*$ such that

$$\begin{aligned} \nu_{u,v} &= h(v)\,\beta_{u,v} \text{ for all } u, v \in G^{(0)} \\ h(r(x))\,\Delta(x) &= \delta(x)\,h(d(x)) \text{ for all } x \in G. \end{aligned}$$

In [3] (p. 84). we have constructed $\delta : G \to \mathbf{R}_+^*$ and $\left\{\beta_{u,v},\, u, v \in G^{(0)}\right\}$ with the above properties and such that

$$\sup_{u,v} \beta_{u,v}(K) < \infty,\ \sup_{x \in K} \delta(x) < \infty \text{ for all compact set } K \subset G.$$

We sketch that construction. We need the following lemma :

Lemma 1. *Let X and Y be metric spaces and let $f : X \to Y$. Let A be a σ-compact subset of X, and $K_1, K_2, ..., K_n, ...$ be a sequence of compact sets whose union is A. If $f|_{K_n}$ is continuous for each n, then there is a Borel function $g : f(A) \to A$ such that $g(f(K_n)) \subset K_n$ for each n and $f(g(y)) = y$ for all $y \in f(A)$.*

Lemma 1 is a slight reformulation (according to [8, Lemma 4.12. p. 99]), based on a result of Federer and Morse [5], of a lemma proved by Mackey (Lemma 1.1 [6]). We call the function g in the Lemma 1 a regular Borel cross section of f. let e be a unit in $G^{(0)}$. Applying Lemma 1 to the continuous surjection $d : G^e \to G^{(0)}$, it follows that there is a regular Borel cross section $\sigma : G^{(0)} \to G^e$. By Theorem 2.2 B/p. 8 [7] or Corollary 5.7 [2], if G is a transitive locally compact second countable groupoid, then the application $d : G^u \to G^{(0)}$ is open. Hence for any compact subset K of $G^{(0)}$ there is a compact subset L in G^u such that K is contained in $d(L)$. Thus the closure of $\sigma(K)$ is a compact set for all compact sets K.

Let U_0 be a closed symmetric d-relatively compact neighborhood of $G^{(0)}$, U be an open symmetric d-relatively compact neighborhood of $G^{(0)}$ such that $U_0 \subset U$, and $f_0 : G \to [0,\, 1]$ be a continuous function with $supp(f) \subset U$ and such that $f(x) = 1$ for all $x \in U_0$. Choose μ_e a (left) Haar measure on the locally compact group G_e^e such that

$$\int f_0(y)\, d\mu_e(y) = 1.$$

Define $\beta_{u,v}$ and δ by

$$\begin{aligned} \int f(y)\,\beta_{u,v}(y) &= \int f\left(\sigma(u)^{-1} y \sigma(v)\right) d\mu_e(y), \text{ for all } f \geq 0 \text{ Borel} \\ \delta(y) &= \Delta_e\left(\sigma(r(y))\, y \sigma(d(y))^{-1}\right), \text{ for all } y \in G \end{aligned}$$

where Δ_e is the modular function on the locally compact group G_e^e. Using the same argument as in [3] (p. 85), we can prove that the function h (the connection between $\left\{\beta_{u,v},\, u, v \in G^{(0)}\right\}$ and $\left\{v_{u,v},\, u, v \in G^{(0)}\right\}$) is μ-integrable on the compact subsets of $G^{(0)}$, i.e. for all compact subsets K of $G^{(0)}$

$$\int 1_K(v)\, h(v)\, d\mu(v) < \infty$$

Remark 2. *Let e be a unit. Let us consider the trivial groupoid $G^{(0)} \times G_e^e \times G^{(0)}$. The topology of $G^{(0)} \times G_e^e \times G^{(0)}$ is the product topology, and the operations are*

$$\begin{aligned} (u, x, v)(v, y, w) &= (u, xy, w) \\ (u, x, v)^{-1} &= (v, x, u). \end{aligned}$$

Under this structure $G^{(0)} \times G_e^e \times G^{(0)}$ is a locally compact second countable groupoid. The system of measures $\left\{\varepsilon_u \times \mu_e \times h \cdot \mu,\, u \in G^{(0)}\right\}$ is Haar system on this groupoid.

It is not hard to prove that the C^*-algebra of $G^{(0)} \times G_e^e \times G^{(0)}$ is isomorphic to $C^*(G_e^e) \otimes \mathcal{K}\left(L^2(\mu)\right)$.

Proposition 3. *Let G be a locally compact second countable transitive groupoid. Let e be a unit and $\sigma : G^{(0)} \to G^e$ be a regular Borel cross section of $d : G^e \to G^{(0)}$. Then $\phi : G \to G^{(0)} \times G_e^e \times G^{(0)}$ defined by*

$$\phi(x) = \left(r(x), \sigma(r(x))\, x\, \sigma(d(x))^{-1}, d(x)\right)$$

is a Borel isomorphism which transport the Haar system of G into a Haar system of $G^{(0)} \times G_e^e \times G^{(0)}$ of the form $\left\{\varepsilon_u \times \mu_e \times \lambda,\ u \in G^{(0)}\right\}$, where ε_u is the unit point mass at $u \in G^{(0)}$, μ_e is a Haar measure on G_e^e, and λ is a suitable Radon measure on $G^{(0)}$.

Proof. Straightforward calculation. The Haar system $\left\{\nu^u,\ u \in G^{(0)}\right\}$ is carried to $\left\{\varepsilon_u \times \mu_e \times h \cdot \mu,\ u \in G^{(0)}\right\}$. □

3. Isomorphism for transitive groupoid C^*-algebras

The following proposition was proved in [3](Proposition 4/p. 82, Proposition 5/p. 86)

Proposition 4. *With the notation of preceding section, let $\nu_0 = \Delta^{-\frac{1}{2}}\mu$ and $f \in L^1(G, \nu_0)$ such that*

$$\int\int\left(\int |f(x)|\,\Delta(x)^{-\frac{1}{2}}\, d\nu_{u,v}(x)\right)^2 d\mu(v)\, d\mu(u) < \infty.$$

Then there is a sequence $(f_n)_n$, in $C_c(G)$ such that

$$\lim_n \|f - f_n\|_{II} = 0.$$

where for any $g \in L^1(G, \nu_0)$, $\|g\|_{II}$ is defined by

$$\|g\|_{II} = \sup\left\{\int |g(x)\, j(d(x))\, k(r(x))|\, d\nu_0(x),\ \int |j|^2\, d\mu = \int |k|^2\, d\mu = 1\right\}.$$

Proposition 5. *If $f \in L^1(G, \nu_0)$ and*

$$\int\int\left(\int |f(x)|\,\Delta(x)^{-\frac{1}{2}}\, d\nu_{u,v}(x)\right)^2 d\mu(v)\, d\mu(u) < \infty.$$

then f can be viewed as an element in $C^*(G)$.

Remark 6. *In particular, any function in $\mathcal{B}_c(G)$, the space of compactly supported Borel bounded function on G, can be viewed as an element in $C^*(G)$.*

Theorem 7. *Let G be a locally compact second countable transitive groupoid. Let $\left\{\nu^u, u \in G^{(0)}\right\}$ be a Haar system on G. Let e be a unit and let us endow the trivial groupoid $G^{(0)} \times G_e^e \times G^{(0)}$ with the Haar system $\left\{\varepsilon_u \times \mu_e \times h\mu,\ u \in G^{(0)}\right\}$, where μ_e and h are chosen as in Section 2. Then the C^*-algebra of G associated with the Haar system $\left\{\nu^u, u \in G^{(0)}\right\}$ and the C^*-algebra of $G^{(0)} \times G_e^e \times G^{(0)}$ associated with the Haar system $\left\{\varepsilon_u \times \mu_e \times h\mu,\ u \in G^{(0)}\right\}$ are $*$-isomorphic.*

Proof. Let ϕ be the groupoid isomorphism defined in Proposition 3. Any nondegenerate representation of $C_c(G)$ is equivalent with the integrated form of a representation of G (Theorem 1.21 /pg.65 [10], or Theorem 3.29 /pg.74 [8]). The same thing is true for $G^{(0)} \times G_e^e \times G^{(0)}$. Since $\phi : G \to G^{(0)} \times G_e^e \times G^{(0)}$ is a Borel groupoid isomorphism, $L \to L \circ \phi$ is an one to one correspondence between the representation of the two groupoids Also ϕ carry the Haar system$\left\{\nu^u, u \in G^{(0)}\right\}$ into the Haar system $\left\{\varepsilon_u \times \mu_e \times h\mu,\ u \in G^{(0)}\right\}$.Therefore $\Phi : B_c\left(G^{(0)} \times G_e^e \times G^{(0)}\right) \to \mathcal{B}_c(G)$ defined by

$$\Phi(f) = f \circ \phi$$

is a *-isomorphism which can be extended to the $C^*\left(G^{(0)} \times G_e^e \times G^{(0)}\right)$. □

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A generalization of the Hyers-Ulam-Rassias stability of a cubic functional equation

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Abstract. In this paper the the stability of the cubic functional equation $f(x+2y)+3f(x)=3f(x+y)+f(x-y)+6f(y)$ in the line of Hyers, Ulam, Rassias and Găvruţa is investigated.

Keywords: Cubic functional equation, Hyers-Ulam-Rassias stability

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1. Introduction

The stability problem of functional equations originated from a question of S. Ulam concerning the stability of a group homomorphism:

Let $(G_1,\circ)$ be a group and let $(G_2,*)$ *be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if $f:G_1\to G_2$ satisfies*

$$d(f(x\circ y),f(x)*f(y))\leq\delta,\ \text{for all } x,y\in G_1,$$

then there exists a homomorphism $h:G_1\to G_2$ with

$$d(f(x),h(x))\leq\varepsilon,\ \text{for all } x\in G_1?$$

D. H. Hyers [4] gave a first affirmative answer to the Ulam's question, for Banach spaces. *Let $f:E_1\to E_2$ be a mapping, where E_1* and *E_2 are Banach spaces, such that*

$$\|f(x+y)-f(x)-f(y)\|\leq\delta,$$

for all $x,y\in E_1$ and for some $\delta>0$. Then there exists a unique additive mapping $T:E_1\to E_2$ satisfying

$$\|f(x)-T(x)\|\leq\delta,\ \text{for all } x\in E_1.$$

Moreover, if $f(tx)$is continuous in t for each fixed x, then the mapping T is linear.

In 1978, Th. M. Rassias [9] proved the more general result:

Let $f:E_1\to E_2$ be a mapping, where E_1 is a real normed space and E_2 is a Banach space. Assume that there exist $\delta>0$ and $p\in[0,1)$ such that

$$\|f(x+y)-f(x)-f(y)\|\leq\delta\cdot\left(\|x\|^p+\|y\|^p\right),\text{for all } x,y\in E_1.$$

Then there exists a unique additive mapping $T:E_1\to E_2$such that

$$\|f(x)-T(x)\|\leq\frac{2\,\delta}{2-2^p}\|x\|^p,\ \text{for all } x\in E_1.$$

Moreover, if $f(tx)$ is continuous in t for each fixed $x\in E_1$, then the mapping T is linear.

Further generalizations of the above results are obtained by P. Găvruţa [1].

Let G be an abelian group, let $(E,\|\cdot\|)$ be a Banach space and let $\varphi:G\times G\to[0,\infty)$ be a mapping such that

$$\Phi(x,y):=\sum_{k=0}^{\infty}2^{-(k+1)}\varphi(2^kx,2^ky)<\infty,\text{for all } x,y\in G.$$

If $f:G\to E$ satisfies

$$\|f(x+y)-f(x)-f(y)\|\leq\varphi(x,y),\ \text{for all } x,y\in G,$$

then there exists a unique mapping $T : G \to E$ with the properties

$$T(x+y) = T(x) + T(y), \text{ for all } x, y \in G$$

and

$$\|f(x) - T(x)\| \leq \Phi(x, x), \text{ for all } x \in G.$$

For other approaches concerning the Hyers-Ulam-Rassias stability see also the books [5] and [7].

In [8] J. M. Rassias introduced a new definition of a cubic mappings:

Definition 1. *Let X and Y two given vector spaces. A mapping $C : X \to Y$ is called cubic if C satisfies the functional equation*

$$f(x+2y) + 3f(x) - 3f(x+y) - f(x-y) - 6f(y) = 0, \tag{1}$$

for all $x, y \in X$.

In [8] he proved that a cubic mapping $C : X \to Y$, satisfying functional equation (1), has the properties:
C is an odd mapping, hence equation $C(-x) = -C(x)$ holds for all $x \in X$;

$$C(x) = \frac{1}{8^n} C(2^n x), \quad \text{for all } x \in X \text{ and for all } n \in \mathbb{N}. \tag{2}$$

The next Hyers-Ulam stability theorem for the functional equation (1) is proved in [8]:

Theorem 2. *Let X be a normed linear space and let Y be a real complete normed linear space. If the mapping $f : X \to Y$ satisfies the inequality*

$$||f(x+2y) + 3f(x) - 3f(x+y) - f(x-y) - 6f(y)|| \leq c,$$

for all $x, y \in X$, (with a constant c independent by x and y), then there exists a unique cubic mapping $C : X \to Y$, which is defined by the limit

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$$

for all $x \in X$, and satisfies the relation

$$||C(x) - f(x)|| \leq \frac{11}{42} c, \quad \text{for all } x \in X.$$

Recently, P. Găvruţa and L. Cădariu [2], studied a modified Hyers-Ulam-Rassias stability for the functional equation (1). They generalized the previous result of J.M. Rassias in the following form:

Theorem 3. *Let G be an abelian group, let X be a Banach space and let $\varphi : G \times G \to [0, \infty)$ be a mapping such that*

$$\Phi(x, y) := \sum_{n=0}^{\infty} \frac{1}{8^n} \varphi(2^n x, 2^n y) < \infty,$$

for all $x, y \in G$.

If $f : G \to X$ is a mapping which satisfies

$$||f(x+2y) + 3f(x) - 3f(x+y) - f(x-y) - 6f(y)|| \leq \varphi(x, y),$$

for all $x, y \in X$, then there exists a unique cubic mapping $C : G \to X$, which is defined by the limit

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n} f(2^n x),$$

for all $x \in G$ such that the relation

$$||C(x) - f(x)|| \leq \frac{1}{48}\Phi(x, -x) + \frac{5}{48}\Phi(0, x) + \frac{1}{14}\varphi(0, 0) \tag{3}$$

holds for all $x \in G$.

In [2], as a particular case of Theorem 3, is obtained a stability theorem for the functional equation (1) if the control function φ is given by $\varphi(x,y) := ||x||^p + ||y||^p$, where $0 < p < 3$.

In the present paper, another stability results in the sense of P. Găvruţă [1] (modified Hyers-Ulam-Rasiass stability) for the cubic functional equation (1) is studied. It is also proved, that functional equation (1) remains stable if $\varphi(x,y) := ||x||^p + ||y||^p$, for $3 < p$, but it is not stable in the sense of Hyers-Ulam-Rassias for $p = 3$. For the proof of this last result we use the next theorem which appears in the A. Gilanyi's paper [3].

Theorem 4. *Let n be a positive integer, let ε be a positive real number and let*

$$\varepsilon^* = \frac{\varepsilon}{2^n(2^n + n!)n^n}.$$

Consider the mapping $\varphi : \mathbb{R} \to \mathbb{R}$

$$\varphi(x) = \begin{cases} n^n\varepsilon^* , & \text{if } x \geq n \\ \varepsilon^* x^n , & \text{if } -n < x < n \\ (-1)^n n^n \varepsilon^*, & \text{if } x \leq n \end{cases}$$

and, for a fixed integer $l \geq 2$, define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{m=0}^{\infty} \frac{\varphi(l^m x)}{l^{mn}}, \text{ for all } x \in \mathbb{R}.$$

For this function we have

$$|\Delta_y^n f(x) - n! f(y)| \leq (|x|^n + |y|^n), \text{ for all } x, y \in \mathbb{R},$$

but there does not exist a real number $c = (n, \alpha)$ for which there exists a monomial function $g : \mathbb{R} \to \mathbb{R}$ of degree n such that

$$||f(x) - g(x)|| \leq c\varepsilon|x|^n, \text{ for all } x \in \mathbb{R}.$$

The difference operator Δ, which appears in the above theorem, is defined for a mapping f from a linear space X into Y and for a positive integer n, by

$$\Delta_y^1 f(x) := f(x+y) - f(x), \text{ for all } x, y \in X,$$

and

$$\Delta_y^{n+1} f(x) = \Delta_y^1 f(x) \Delta_y^n f(x), \text{ for all } x, y \in X.$$

Definition 5. *Let X and Y be two given vector spaces. A mapping $f : X \to Y$ is called amonomial function of degree n if the functional equation*

$$\Delta_y^n f(x) - n! f(y) = 0,$$

holds for all $x, y \in X$.

Remark 6. *If $n = 3$, then $\Delta_y^3 f(x) = f(x+3y) - 3f(x+2y) + 3f(x+y) - f(x)$, and, therefor,*

$$\Delta_y^3 f(x) - 3! f(y) = f(x+3y) - 3f(x+2y) + 3f(x+y) - f(x) - 6f(y), \text{ for all } x, y \in X.$$

If in the last relation we replace x by x-y, we obtain that the notion of monomial function of degree 3 is equivalent to the notion of cubic functional equation in the sense of Definition 1.

2. The modified Hyers-Ulam-Rassias stability of cubic functional equation (1)

The general Hyers-Ulam-Rassias stability of the cubic functional equation (1) is presented.

Let $(G, +)$ be an Abelian group such that, for any $x \in G$, there exists a unique $a \in G$ with the property $x = 2a$. The unique element a of G with this property is denoted by $\frac{x}{2}$.

We consider a Banach space $(X, ||\cdot||)$ and a mapping $\varphi : G \times G \to [0, \infty)$ such that

$$\Phi(x,y) := \sum_{n=0}^{\infty} 2^{3n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty, \tag{4}$$

for all $x, y \in G$. We can formulate our main result:

Theorem 7. *Let $f : G \to X$, be such that*

$$||f(x+2y)+3f(x)-3f(x+y)-f(x-y)-6f(y)|| \leq \varphi(x,y). \tag{5}$$

for all $x, y \in G$.

Then there exists a unique cubic mapping $C : G \to X$, which is defined by the limit

$$C(x) := \lim_{n\to\infty} 2^{3n} f\left(\frac{x}{2^n}\right),$$

for each $x \in G$ such that the relation

$$||f(x)-C(x)|| \leq \frac{1}{6}\Phi\left(\frac{x}{2},-\frac{x}{2}\right)+\frac{5}{6}\Phi\left(0,\frac{x}{2}\right), \tag{6}$$

holds for all $x \in G$.

(i) Proof of existence. In (5) we take $x = y = 0$. It follows that $||6f(0)|| \leq \varphi(0,0)$. From (4) we have that $\varphi(0,0) = 0$, therefore it follows that $f(0) = 0$.

In relation (5) we replace x by $\frac{x}{2}$ and y by $-\frac{x}{2}$, respectively, to obtain

$$\left|\left|f\left(-\frac{x}{2}\right)+3f\left(\frac{x}{2}\right)-3f(0)-f(x)-6f\left(-\frac{x}{2}\right)\right|\right| \leq \varphi\left(\frac{x}{2},-\frac{x}{2}\right),$$

hence

$$\left|\left|-5f\left(-\frac{x}{2}\right)+3f\left(\frac{x}{2}\right)-f(x)\right|\right| \leq \varphi\left(\frac{x}{2},-\frac{x}{2}\right), \tag{7}$$

for all $x \in G$.

In (5) we put $x = 0$ and $y = \frac{x}{2}$, we obtain

$$\left|\left|f(x)+3f(0)-3f\left(\frac{x}{2}\right)-f\left(-\frac{x}{2}\right)-6f\left(\frac{x}{2}\right)\right|\right| \leq \varphi\left(0,\frac{x}{2}\right),$$

hence

$$\left|\left|5f\left(-\frac{x}{2}\right)+45f\left(\frac{x}{2}\right)-5f(x)\right|\right| \leq 5\varphi\left(0,\frac{x}{2}\right), \tag{8}$$

for all $x \in G$.

From (7), (8), taking into acount the triangle inequality it follows that

$$\left|\left|-6f(x)+48\left(\frac{x}{2}\right)\right|\right| \leq \varphi\left(\frac{x}{2},-\frac{x}{2}\right)+5\varphi\left(0,\frac{x}{2}\right),$$

hence

$$\left|\left|2^3 f\left(\frac{x}{2}\right)-f(x)\right|\right| \leq \frac{1}{6}\varphi\left(\frac{x}{2},-\frac{x}{2}\right)+\frac{5}{6}\varphi\left(0,\frac{x}{2}\right) \tag{9}$$

for all $x \in G$.

Denote

$$\psi(x) := \frac{1}{6}\varphi\left(\frac{x}{2},-\frac{x}{2}\right)+\frac{5}{6}\varphi\left(0,\frac{x}{2}\right),$$

for each $x \in G$.

We will prove, by mathematical induction on n, that

$$\left|\left|2^{3n}\left(\frac{x}{2^n}\right)-f(x)\right|\right| \leq \psi(x)+2^3\psi\left(\frac{x}{2}\right)+...+2^{3(n-1)}\psi\left(\frac{x}{2^{n-1}}\right) \tag{10}$$

By (9), it is obvious that (10) holds for $n = 1$.

Suppose that (10) holds for $n \in \mathbb{N}$. We prove that (10) is true for $n+1$. Replacing x by $\frac{x}{2}$ into (10) and using the triangle inequality, it follows:

$$\begin{aligned}\left|\left|2^{3(n+1)} f\left(\frac{x}{2^{n+1}}\right)-f(x)\right|\right| &\leq \left|\left|2^{3(n+1)}\left(\frac{x}{2^{n+1}}\right)-2^3 f\left(\frac{x}{2}\right)\right|\right|+\left|\left|2^3 f\left(\frac{x}{2}\right)-f(x)\right|\right| \leq \\ &\leq \psi(x)+2^3\psi\left(\frac{x}{2}\right)+...+2^{3n}\psi\left(\frac{x}{2^n}\right),\end{aligned}$$

hence relation (10) holds for all $n \in \mathbb{N}$.

We claim that the sequence $\{2^{3n}f(2^{-n}x)\}_{n\geq 0}$ is a Cauchy sequence. Indeed, for $m > n > 0$, using relation (10), it follows

$$\begin{aligned}\left|\left|2^{3m}f\left(\frac{x}{2^m}\right) - 2^{3n}f\left(\frac{x}{2^n}\right)\right|\right| &= 2^{3n}\left|\left|2^{3(m-n)}f\left(\frac{x}{2^{(m-n)}\cdot 2^n}\right) - f\left(\frac{x}{2^n}\right)\right|\right| \leq \\ &\leq 2^{3n}\left[\psi\left(\frac{x}{2^n}\right) + 2^3\psi\left(\frac{x}{2^{n+1}}\right) + ... + 2^{3(m-n-1)}\psi\left(\frac{x}{2^n}\cdot\frac{1}{2^{m-n-1}}\right)\right] = \\ &= 2^{3n}\psi\left(\frac{x}{2^n}\right) + 2^{3(n+1)}\psi\left(\frac{x}{2^{n+1}}\right) + ... + 2^{3(m-1)}\psi\left(\frac{x}{2^{m-1}}\right) = \\ &= \sum_{p=n}^{m-1} 2^{3p}\psi\left(\frac{x}{2^p}\right),\end{aligned}$$

for all all $x \in G$.

Taking the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty}\left|\left|2^{3m}f\left(\frac{x}{2^m}\right) - 2^{3n}f\left(\frac{x}{2^n}\right)\right|\right| = 0.$$

Since X is a Banach space, it follows that the sequence $\{2^{3n}f(2^{-n}x)\}_{n\geq 0}$ converges. We define

$$C(x) := \lim_{n\to\infty} 2^{3n}f\left(\frac{x}{2^n}\right),$$

for each x in G.

From (10) we obtain

$$||C(x) - f(x)|| \leq \sum_{n=0}^{\infty} 2^{3n}\psi\left(\frac{x}{2^n}\right),$$

hence

$$||C(x) - f(x)|| \leq \frac{1}{6}\Phi\left(\frac{x}{2}, -\frac{x}{2}\right) + \frac{5}{6}\Phi\left(0, \frac{x}{2}\right), \tag{11}$$

for all x in G.

We claim that C is a cubic mapping in G. Indeed, replacing x by $2^{-n}x$ and y by $2^{-n}y$ respectively, in the cubic functional inequality (5) and multiplicating this relation by 2^{3n}, it follows

$$\begin{aligned}||2^{3n}f\left(2^{-n}x + 2\cdot 2^{-n}y\right) &+ 3\cdot 2^{3n}f\left(2^{-n}x\right) - 3\cdot 2^{3n}f\left(2^{-n}x + 2^{-n}y\right) - \\ &- 2^{3n}f\left(2^n x - 2^{-n}y\right) - 6\cdot 2^{3n}f\left(2^{-n}y\right)|| \leq 2^{3n}\varphi(2^{-n}x, 2^{-n}y),\end{aligned}$$

for all $x, y \in G$, and all $n \in \mathbb{N}$.

Taking the limit as $n \to \infty$, we obtain,

$$C(x + 2y) + 3C(x) - 3C(x + y) - C(x - y) - 6C(y) = 0$$

for all $x, y \in G$, therefor C is a cubic mapping.

(ii) Proof of uniqueness.

Let $C_1 : G \to X$ be another cubic mappings satisfying the cubic functional equation (1), such that inequality

$$||C_1(x) - f(x)|| \leq \frac{1}{6}\Phi\left(\frac{x}{2}, -\frac{x}{2}\right) + \frac{5}{6}\Phi\left(0, \frac{x}{2}\right), \tag{12}$$

holds for all $x \in G$. From relation (2), it follows

$$C(x) := 2^{3m}C\left(\frac{x}{2^m}\right) \text{ and } C_1(x) := 2^{3m}C_1\left(\frac{x}{2^m}\right),$$

for all $x \in X$, and all $m \in \mathbb{N}$.

Using the triangle inequality, the relations (6), (10) and (12), it follows

$$\begin{aligned}
||C(x)-C_1(x)|| &= ||2^{3m}C(2^{-m}x)-2^{3m}C_1(2^{-m}x)|| \leq \\
&\leq 2^{3m}||C(2^{-m}x)-f(2^{-m}x)||+2^{3m}||f(2^{-m}x)-C_1(2^{-m}x)|| \leq \\
&\leq 2\cdot\frac{1}{6}2^{3m}\cdot\Phi\left(\frac{2^{-m}x}{2},-\frac{2^{-m}x}{2}\right)+2\cdot\frac{5}{6}\cdot 2^{3m}\cdot\Phi\left(0,\frac{2^{-m}x}{2}\right)= \\
&= \frac{1}{3}\cdot 2^{3m}\cdot\sum_{n=0}^{\infty}2^{3n}\varphi\left(\frac{x}{2^{(n+m+1)}},-\frac{x}{2^{(n+m+1)}}\right)+2^{3m}\cdot\frac{5}{3}\cdot\sum_{n=0}^{\infty}2^{3n}\varphi\left(0,\frac{x}{2^{(n+m+1)}}\right)= \\
&= \frac{1}{3\cdot 2^3}\sum_{n=0}^{\infty}2^{3(n+m+1)}\varphi\left(\frac{x}{2^{(n+m+1)}},-\frac{x}{2^{(n+m+1)}}\right)+ \\
&+ \frac{5}{3\cdot 2^3}\sum_{n=0}^{\infty}2^{3(n+m+1)}\varphi\left(0,\frac{x}{2^{(n+m+1)}}\right)= \\
&= \frac{1}{24}\sum_{p=m+1}^{\infty}2^{3p}\varphi\left(\frac{x}{2^p},-\frac{x}{2^p}\right)+\frac{5}{24}\sum_{p=m+1}^{\infty}2^{3p}\varphi\left(0,\frac{x}{2^p}\right),
\end{aligned}$$

for all $x\in G$, and all $m\in\mathbb{N}$.

Therefore, from the above inequality, taking the limit as $m\to\infty$, we obtain

$$||C(x)-C_1(x)||=0,$$

for all $x\in G$.

It follows that C coincides with C_1. This completes the proof of the theorem. □

Corollary 8. *Let X be a normed linear space and let Y be a real complete normed linear space. Let be $\varepsilon\geq 0$ and $p\in\mathbb{R}$, $p>3$. If $f:X\to Y$ is a mapping such that the following inequality*

$$||f(x+2y)+3f(x)-3f(x+y)-f(x-y)-6f(y)||\leq\varepsilon\cdot(||x||^p+||y||^p)$$

holds for all $x,y\in X$, then there exists a unique cubic mapping $C:X\to Y$, defined by the limit

$$C(x):=\lim_{n\to\infty}2^{3n}f\left(\frac{x}{2^n}\right),$$

for all $x\in X$, and for all $n\in\mathbb{N}$, such that

$$||C(x)-f(x)||\leq\frac{7\varepsilon}{6}\cdot||x||^p\cdot\frac{1}{2^p-8}$$

for all $x\in X$.

Proof. If in Theorem 7 we take $\varphi(x,y)=\varepsilon\cdot(||x||^p+||y||^p)$, $p>3$ it results:

$$\psi(x)=\frac{1}{6}\varphi\left(\frac{x}{2},-\frac{x}{2}\right)+\frac{5}{6}\varphi\left(0,\frac{x}{2}\right)=\frac{\varepsilon}{6}\left(\left|\left|\frac{x}{2}\right|\right|^p+\left|\left|-\frac{x}{2}\right|\right|^p\right)+\frac{5\varepsilon}{6}\left|\left|\frac{x}{2}\right|\right|^p=\frac{7\varepsilon}{6}\left|\left|\frac{x}{2}\right|\right|^p.$$

Therefor, there exists a unique cubic mapping C, such that

$$\begin{aligned}
||C(x)-f(x)|| &\leq \frac{1}{6}\Phi\left(\frac{x}{2},-\frac{x}{2}\right)+\frac{5}{6}\Phi\left(0,\frac{x}{2}\right)=\sum_{n=0}^{\infty}2^{3n}\psi\left(\frac{x}{2^n}\right)=\frac{7\varepsilon}{6}\sum_{n=0}^{\infty}2^{3n}\left|\left|\frac{x}{2^{n+1}}\right|\right|^p= \\
&= \frac{7\varepsilon}{6}\left|\left|\frac{x}{2}\right|\right|^p\sum_{n=0}^{\infty}2^{n(3-p)}=\frac{7\varepsilon}{6}\frac{||x||^p}{2^p}\cdot\frac{1}{1-2^{3-p}}=\frac{7\varepsilon}{6}\cdot||x||^p\cdot\frac{1}{2^p-8}
\end{aligned}$$

for all x in G. □

The next theorem shows that the cubic functional equation is not stable in the sense of Hyers-Ulam-Rassias if $p=3$.

Theorem 9. *Let ε be a positive real number and consider the mapping $\varphi : \mathbb{R} \to \mathbb{R}$*

$$\varphi(x) = \begin{cases} \frac{\varepsilon}{112}\ , & \text{if } x \geq 3 \\ \frac{\varepsilon}{3024}x^3, & \text{if } -3 < x < 3 \\ -\frac{\varepsilon}{112}\ , & \text{if } x \leq 3 \end{cases}$$

For a fixed integer $l \geq 2$, we define a function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{m=0}^{\infty} \frac{\varphi(l^m x)}{l^{3m}},\ \text{for all } x \in \mathbb{R}.$$

For this function we have

$$||f(x+2y) + 3f(x) - 3f(x+y) - f(x-y) - 6f(y)|| \leq \varepsilon \left(|x|^3 + |y|^3\right),\ \text{for all } x, y \in \mathbb{R},$$

but there does not exist a real number k for which there exists a cubic function $C : \mathbb{R} \to \mathbb{R}$ such that

$$||f(x) - C(x)|| \leq k\varepsilon|x|^3,\ \text{for all } x \in \mathbb{R}.$$

Proof. Using Remark 1.6. it is obviously that this theorem is a particular case of Theorem 4 for $n = 3$. □

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A note on the Cauchy-Riemann conditions

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Abstract. A special class of smooth functions, called strongly polyholomorphic functions, is studied. This is a generalization of the holomorphic functions for the 3-dimensional case or higher, introduced by a natural extension of the Cauchy-Riemann conditions. This functions are conformal maps of the first kind, which is a nice geometric property. It is shown that for $\mathbb{R}^3$, this functions form a 10-dimensional linear space over the reals, with respect to the addition .

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Definition 1 ([1]). *Let G be a nonempty open set in $\mathbb{R}^n$, $n \geq 2$, and $f = (f_1, f_2, \ldots, f_n) : G \to \mathbb{R}^n$ a differentiable function on G. The function f is said to be:*

(a) *weakly polyholomorphic on G if there exists a pair (f_i, f_j), $i \neq j$, such that the Cauchy-Riemann conditions*

$$\text{(1)} \qquad \begin{cases} \frac{\partial f_i}{\partial x_i} = \frac{\partial f_j}{\partial x_j}, \\ \frac{\partial f_i}{\partial x_j} = -\frac{\partial f_j}{\partial x_i} \end{cases}$$

hold on G;

(b) *polyholomorphic on G if there exists i, $1 \leq i \leq n$, such that the relations (1) take place for every $j \neq i$, $1 \leq j \leq n$, on G;*

(c) *strongly polyholomorphic on G if the relations (1) are satisfied for every pair (f_i, f_j), $i \neq j$, $1 \leq i, j \leq n$, on G.*

Remark 2 ([1]). *For $n = 2$, the class of weakly polyholomorphic, polyholomorphic and strongly polyholomorphic functions coincides with the class of holomorphic functions.*

Example 3 ([1]). *A linear function $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(x) = Ax$ is strongly polyholomorphic if and only if A is antisymmetric and with equal elements on the main diagonal.*

Example 4 ([1]). *The function $f : \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$f(x_1, x_2, \ldots, x_n) = \left(x_1^2 - x_2^2 - \cdots - x_n^2, 2x_1x_2, \ldots, 2x_1x_n\right)$$

is strongly polyholomorphic on $\mathbb{R}^n$.

The strongly polyholomorphic functions are smooth functions, as proved in [1].

Proposition 5 ([1]). *If $f : G \to \mathbb{R}^n$ is a strongly polyholomorphic function on G then $f \in C^\infty(G)$.*

The strongly polyholomorphic functions have a nice geometric property. Since the holomorphic functions are conformal maps of the first kind (preserve the angle between any two smooth curves which intersect, see [2, p. 40-42] for more details), in [3] it has been proved that the same property holds for the strongly polyholomorphic functions in $\mathbb{R}^3$.

Unfortunately, the class of strongly polyholomorphic functions is not large. This is a consequence of the fact that the conditions are very restrictive. This fact has been proved in [3], where the following representation result is given for the class of strongly polyholomorphic functions in $\mathbb{R}^3$.

Theorem 6 ([3]). *Let G be a domain in $\mathbb{R}^3$. A function $F = (f, g, h) : G \to \mathbb{R}^3$ is strongly polyholomorphic on G iff*

$$f(x,y,z) = \beta\left(y^2 + z^2 - x^2\right) + P(y,z)x + Q(x,y,z),$$
$$g(x,y,z) = -\int_{x_0}^{x} (2\beta y + \frac{\partial P}{\partial y}x + \frac{\partial Q}{\partial y})\,\mathrm{d}\,x + \varphi(y,z),$$
$$h(x,y,z) = -\int_{x_0}^{x} (2\beta z + \frac{\partial P}{\partial z}x + \frac{\partial Q}{\partial z})\,\mathrm{d}\,x + \psi(y,z),$$

where $P(y,z)$ and $Q(x,y,z)$ are arbitrary polynomial functions with $\deg P \leq 1$, $\deg Q \leq 1$ and

$$\varphi(y,z) = \frac{1}{2}\left(y^2 - z^2\right)\frac{\partial P}{\partial y} + \left(P(0,z) + \frac{\partial Q}{\partial x} - 2\beta x_0\right)y - qz + k_1$$
$$\psi(y,z) = \frac{1}{2}\left(z^2 - y^2\right)\frac{\partial Q}{\partial z} + \left(P(y,0) + \frac{\partial Q}{\partial x} - 2\beta x_0\right)z + qy + k_2$$

where $\beta, q, k_1, k_2 \in \mathbb{R}$.

This result gives a full representation of the strongly polyholomorphic functions in the three-dimensional case, but the formulas involved are not so neat and hardly accept an obvious generalization for the $n-$dimensional case.

We shall prove a more clean representation for the strongly polyholomorphic functions in the three-dimensional case, which can be easily extended to higher dimensions.

First of all, let us remark that the class of strongly polyholomorphic functions on a nonempty open set G is a real linear space with respect to the addition, denoted by $(SPH^n(G), \mathbb{R}, +)$ or, shortly, SPH^n.

The main result follows.

Theorem 7. *If G is a domain in $\mathbb{R}^3$, then SPH^3 is a ten$-$dimensional linear space, where a basis $\mathcal{B}$ consists of the following linearly independent function:*

$$(x,y,z) \to \left(x^2 - y^2 - z^2, 2xy, 2yz\right)$$
$$(x,y,z) \to \left(2xy, y^2 - z^2 - x^2, 2yz\right)$$
$$(x,y,z) \to \left(2xz, 2yz, z^2 - x^2 - y^2\right)$$

$$(x,y,z) \to (x,y,z)$$
$$(x,y,z) \to (0,z,-y)$$
$$(x,y,z) \to (-z,0,x)$$
$$(x,y,z) \to (y,-x,0)$$

and

$$(x,y,z) \to (1,0,0)$$
$$(x,y,z) \to (0,1,0)$$
$$(x,y,z) \to (0,0,1).$$

Proof. Let $F = (f,g,h) : G \to \mathbb{R}^3$ be a strongly polyholomorphic function on the domain G and depending on the variables x, y and z. Recall that F is smooth on G and satisfies the following relations on G

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial h}{\partial z}, \tag{2}$$
$$\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x};\ \frac{\partial g}{\partial z} = -\frac{\partial h}{\partial y};\ \frac{\partial h}{\partial x} = -\frac{\partial f}{\partial z}.$$

Starting with these, we obtain the following: $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = -\frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = -\frac{\partial^2 f}{\partial y \partial z}$ implying that

$$\text{(3a)} \qquad \frac{\partial^2 f}{\partial y \partial z} = 0$$

By symmetry, the following relations:

$$\text{(3b)} \qquad \frac{\partial^2 g}{\partial x \partial z} = 0$$
$$\frac{\partial^2 h}{\partial x \partial y} = 0$$

take place too. Again using (2) we obtain that $\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial h}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial^2 h}{\partial x \partial y}\right)$ and, by (3b), the result will be

$$\text{(4a)} \qquad \frac{\partial^3 f}{\partial x^2 \partial y} = 0.$$

By symmetry we also get

$$\text{(4b)} \qquad \frac{\partial^3 f}{\partial x^2 \partial z} = 0.$$

For g and h, the above relations become

$$\text{(4c)} \qquad \frac{\partial^3 g}{\partial y^2 \partial x} = \frac{\partial^3 g}{\partial y^2 \partial z} = 0,$$
$$\frac{\partial^3 h}{\partial z^2 \partial x} = \frac{\partial^3 h}{\partial z^2 \partial x} = 0.$$

Again, using (2) we have that $\frac{\partial^3 f}{\partial x^3} = \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial g}{\partial x}\right) = -\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial f}{\partial y}\right) = -\frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = -\frac{\partial^2}{\partial y^2}\left(\frac{\partial g}{\partial y}\right)$, that is

$$\frac{\partial^3 f}{\partial x^3} = -\frac{\partial^3 g}{\partial y^3}$$

and the symmetrical relations

$$\frac{\partial^3 g}{\partial y^3} = -\frac{\partial^3 h}{\partial z^3},$$
$$\frac{\partial^3 h}{\partial z^3} = -\frac{\partial^3 f}{\partial x^3},$$

which can take place only if

$$\text{(5)} \qquad \frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 g}{\partial y^3} = \frac{\partial^3 h}{\partial z^3} = 0.$$

Consequently from (4a), (4b) and (5)

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial x^2}\right) = 0$$

we have that

$$\nabla\left(\frac{\partial^2 f}{\partial x^2}\right) = 0$$

on G, hence the function $\frac{\partial^2 f}{\partial x^2}$ is constant on G, and, symmetrically, so will be $\frac{\partial^2 g}{\partial y^2}$. Thus in $\mathbb{R}^3$ $\frac{\partial^2 h}{\partial z^2}$, there exists $a = (a_1, a_2, a_3)$ such that

$$\frac{\partial^2 f}{\partial x^2} = a_1,\ \frac{\partial^2 g}{\partial y^2} = a_2,\ \frac{\partial^2 h}{\partial z^2} = a_3 \tag{6}$$

on G.

Moreover, from (2) and (6) we obtain that $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial^2 g}{\partial y^2} = a_2$ and, by symmetry, we have $\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = a_3$. Together with $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = a_1$ we conclude that

$$\nabla\left(\frac{\partial f}{\partial x}\right) = a$$

on G. This takes place iff $\frac{\partial f}{\partial x} - (a_1 x + a_2 y + a_3 z)$ is constant on G, since $\nabla\left(\frac{\partial f}{\partial x} - (a_1 x + a_2 y + a_3 z)\right) = 0$. Thus there is a real constant b_0 such that

$$\frac{\partial f}{\partial x} = a_1 x + a_2 y + a_3 z + b_0 \tag{7a}$$

on G. Using (2), the same equality holds for $\frac{\partial g}{\partial y}$ and $\frac{\partial h}{\partial z}$.

Let us also find $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$. We have the followings

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial^2 g}{\partial y^2} = a_2$, using (2) and (6);

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -a_1$, using (2) and (6);

$\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial z} = 0$, using (3a).

We conclude that

$$\nabla\left(\frac{\partial f}{\partial y}\right) = (a_2, -a_1, 0)$$

on G and, symmetrically,

$$\nabla\left(\frac{\partial f}{\partial z}\right) = (a_3, 0, -a_1).$$

Thus there exist two real constants b_1 and b_2 such that

$$\frac{\partial f}{\partial y} = a_2 x - a_1 y + b_1 \tag{7b}$$

and

$$\frac{\partial f}{\partial z} = a_3 x - a_1 z + b_2 \tag{7c}$$

on G. Symmetric formulas take place for g and h. By direct use of (2) we obtain

$$\frac{\partial g}{\partial x} = -a_2 x + a_1 y - b_1 \tag{8a}$$

and

$$\frac{\partial h}{\partial x} = -a_3 + a_1 z - b_2. \tag{8b}$$

Again by symmetry, we have

$$\nabla\left(\frac{\partial g}{\partial z}\right) = (0, a_3, -a_2),$$

therefore there is a real constant b_3 such that

$$\frac{\partial g}{\partial z} = a_3 y - a_2 z + b_3 \tag{8c}$$

on G; using (2), we also obtain

$$\frac{\partial h}{\partial y} = -a_3 y + a_2 z - b_3. \tag{8d}$$

Let us come back to (7a), (7b), (7c) and finally compute f. It easy to check that the function $\overline{f} : G \to \mathbb{R}^3$ defined by

$$\overline{f}(x,y,z) = \frac{a_1}{2}\left(x^2 - y^2 - z^2\right) + a_2 xy + a_3 xz$$

satisfies

$$\frac{\partial \overline{f}}{\partial x} = a_1 x + a_2 y + a_3 z,$$
$$\frac{\partial \overline{f}}{\partial y} = a_2 x - a_1 y,$$
$$\frac{\partial \overline{f}}{\partial z} = a_3 x - a_1 z,$$

consequently $\nabla\left(f - \overline{f}\right) = (b_0, b_1, b_2)$, which can take place iff $f - \overline{f} - (b_0 x + b_1 y + b_2 z)$ is constant on G. It means that there exists a real constant c_1 such that on G, f has the form

$$f(x,y,z) = \frac{a_1}{2}\left(x^2 - y^2 - z^2\right) + a_2 xy + a_3 xz + b_0 x + b_1 y + b_2 z + c_1. \tag{9a}$$

By using the same arguments as above, we can obtain the expressions for g

$$g(x,y,z) = \frac{a_2}{2}(y^2 - x^2 - z^2) + a_1 xy + a_3 yz - b_1 x + b_0 y + b_3 z + c_2 \tag{9b}$$

and h

$$h(x,y,z) = \frac{a_3}{2}(z^2 - x^2 - y^2) + a_2 yz + a_1 xz - b_2 x - b_3 y + b_0 z + c_3, \tag{9c}$$

where c_2 and c_3 are real constants.

Finally, the expression of F reads

$$F(X) = a_1 P_1(X) + a_2 P_2(X) + a_3 P_3(X) + BX + C, \tag{10}$$

where

$$B = \begin{pmatrix} b_0 & b_1 & b_2 \\ -b_1 & b_0 & b_3 \\ -b_2 & -b_3 & b_0 \end{pmatrix}, C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and

$$P_1(X) = \begin{pmatrix} \frac{x^2 - y^2 - z^2}{2} \\ xy \\ xz \end{pmatrix}, P_2(X) = \begin{pmatrix} yx \\ \frac{y^2 - z^2 - x^2}{2} \\ yz \end{pmatrix}, P_3(X) = \begin{pmatrix} zx \\ zy \\ \frac{z^2 - x^2 - y^2}{2} \end{pmatrix}.$$

Now it is obvious that F depends linearly on the 10 linearly independent functions of $\mathcal{B}$, which concludes the proof. □

Following the ideas from the previous proof, it can be easily shown the more general result

Theorem 8. *If G is a domain in $\mathbb{R}^n$, then SPH^n is a $\frac{(n+1)(n+2)}{2}-$dimensional linear space and the general form of the functions in SPH^n is*

$$f(x) = \langle a, x\rangle x - \frac{1}{2}\langle x, x\rangle a + Bx + c,$$

where a, $c \in \mathbb{R}^n$ and B is a $n \times n$ sqew-symmetric matrix with the elements of the main diagonal equal.

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The volume function variation of one chamber of Panu-Stănescu rotary engine

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Abstract. The paper presents the parameters equation of the inside surface section of the crankase of the piston section and the determination of the volum function variation of one chamber.

1. The geometry of the inner surface of the carcass

The *Panu-Stănescu* rotary engine is a heat rotary piston engine, Romanian letters-patent of engineers Mihai Panu and Gheorghe Stănescu. The engine consists on cylindrical carcass(1) with two different values and antipodal ovalizations. The rotary piston (2) is mounted inside the carcass and includes six box-fire(3) for each chamber. On the external surface of the piston there are six radial channel disposed at sixty degree, in which six obturation bars(4) glides. During the running, the bars are in permanent contact with the surface of carcass due to centrifugal forces.

Let we denote by (ρ_1, φ) the polar coordinates of the point M from external extremity of the bar P_1 when this passes through the first ovalization and we denote by (ρ_2, φ) the polar coordinates of the point M when the bar passes through the second ovalization.

From the triangles $\triangle OO_1P_1$ and $\triangle OO_1M$ we get the relations:

$$R_1^2 = R^2 + e_1^2 - Re_1\ ;\ \ \rho_1^2 - 2e_1 \sin\varphi\rho_1 - (R_1^2 - e_1^2) = 0\ .$$

The equation yields the polar line of the first ovalization :

$$\rho_1(\varphi) = e_1 \sin\varphi + \sqrt{R^2 - Re_1 + e_1^2 \sin^2\varphi} \tag{1}$$

Also, from the triangles $\triangle OO_2P_4$ and $\triangle OO_2M$ we get analogously the polar line of the second ovalization

$$\rho_2(\varphi) = -e_2 \sin\varphi + \sqrt{R^2 - Re_2 + e_2^2 \sin^2\varphi} \tag{2}$$

2. The volume function variation

Between carcass, piston and the bars are formed six chambers with variable volum. When the bar P_1 passes through the first ovalization, inside the chamber $P_1 - P_6$ take place the admission-compression processes and when the bar passes through the second ovalization take place the expansion- evacuation processes.

The volum of a chamber formed between the piston and carcass has a cyclical variation according to φ angle by the formula

$$V(\varphi) = l_p A(\varphi) + V_{ca} \tag{3}$$

where: l_p -the breadth of the piston, $A(\varphi)$ -the plan area of one chamber, V_{ca} -the volum of the combustion chamber.

The φ position of the piston determines the following expressions of the $A(\varphi)$ area:

a) for $\varphi \in \left[0, \frac{\pi}{6}\right]$, we obtain

$$A_1(\varphi) = \iint\limits_D dxdy = \iint\limits_{D'} |J| drd\varphi = \frac{1}{2} \int\limits_{\frac{\pi}{6}}^{\varphi+\frac{\pi}{3}} \left(\rho_1^2(\varphi) - R^2\right) d\varphi$$

b) for $\varphi \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, $\quad A_2(\varphi) = \frac{1}{2} \int\limits_{\varphi}^{\varphi+\frac{\pi}{3}} \left(\rho_1^2(\varphi) - R^2\right) d\varphi$

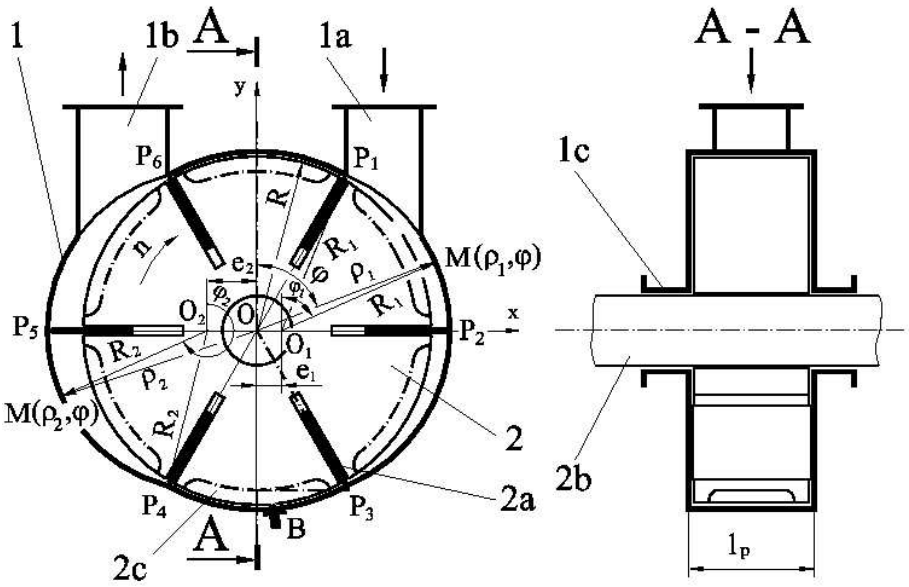

FIGURE 1. Cross section of the engine.

c) for $\varphi \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$, $\quad A_3(\varphi) = \frac{1}{2}\int\limits_{\varphi}^{\frac{5\pi}{6}} \left(\rho_1^2(\varphi) - R^2\right) d\varphi$

d) for $\varphi \in \left[\frac{5\pi}{6}, \frac{7\pi}{6}\right]$, $\quad A_4(\varphi) = \frac{1}{2}\int\limits_{\frac{7\pi}{6}}^{\frac{\pi}{3}+\varphi} \left(\rho_2^2(\varphi) - R^2\right) d\varphi$

e) for $\varphi \in \left[\frac{7\pi}{6}, \frac{3\pi}{2}\right]$, $\quad A_5(\varphi) = \frac{1}{2}\int\limits_{\varphi}^{\varphi+\frac{\pi}{3}} \left(\rho_2^2(\varphi) - R^2\right) d\varphi$

f) for $\varphi \in \left[\frac{3\pi}{2}, \frac{11\pi}{6}\right]$, $\quad A_6(\varphi) = \frac{1}{2}\int\limits_{\varphi}^{\frac{11\pi}{6}} \left(\rho_2^2(\varphi) - R^2\right) d\varphi$

g) for $\varphi \in \left[\frac{11\pi}{6}, 2\pi\right]$, $\quad A_7(\varphi) = \frac{1}{2}\int\limits_{\frac{\pi}{6}}^{\varphi-\frac{10\pi}{6}} \left(\rho_1^2(\varphi) - R^2\right) d\varphi$

The volum function variation $V(\varphi)$ depend continously on φ and it has the branches $V_i(\varphi) = l_p A_i(\varphi) + V_{ca}, \quad i = \overline{1,7}$ with angle φ in the adequate intervals. The maximal volum at admission-compression is obtained for $\varphi = \pi/3$ and the maximal volum at expansion-evacuation is obtained for $\varphi = 4\pi/3$.

We denote by $\varepsilon = V_{max}/V_{min}$ the ratio of compression and we denote by $\delta = V_{d\,max}/V_{min}$ the ratio of expansion, where $V_{d\,max}$ is obtained by the formula

$$V_{d\,max} = l_p A_5\left(\frac{4\pi}{3}\right) + V_{ca} \tag{4}$$

The volum of combustion chamber $V_{ca} = V_{min}$ is calculated by the relation

$$V_{ca} = \frac{1}{\varepsilon - 1} l_p A_2\left(\frac{\pi}{3}\right) \tag{5}$$

By means of (4) and (5) we get

$$\delta = 1 + (\varepsilon - 1)\frac{A_5\left(\frac{4\pi}{3}\right)}{A_2\left(\frac{\pi}{3}\right)} \tag{6}$$

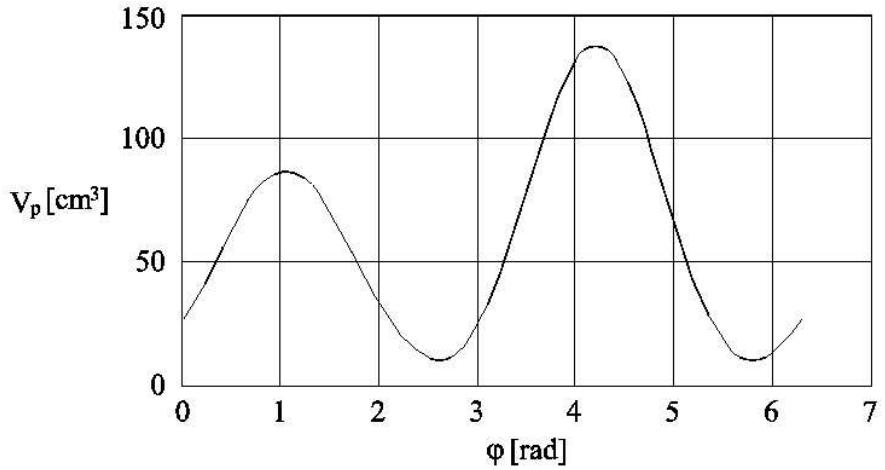

FIGURE 2. Variation of $V(\varphi)$ at one chamber.

At a single rotation of the piston take place six working cycle.

Figure 2 shows the variation of the volum of an rotary engine with following specific parameters: the ratio of compression $\varepsilon = 8,5$, the values of ovalizations $e_1 = 9,25\ mm$, $e_2 = 14,72\ mm$, the radius of the piston $R = 130\ mm$, the breadth of the piston $l_p = 125\ mm$.

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Comparing threads' usage in Linux and Win32 for solving large sparse systems of nonlinear equations

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Abstract. In [2] a method for solving a large sparse nonlinear systems is presented. The method uses threads in a Win32 environment. This paper presents the same method using also threads in Linux and compares both method for speedup and efficiency.

Keywords: Parallel algorithms, iterative methods, sparse nonlinear system

1. Introduction

Solving large nonlinear systems of equations is a hard task. Because most large systems are sparse, one can think to take advantage of the sparsity by solving the system in parallel.

Many methods have been proposed to solve such a system in parallel; most of them do not decompose the original system into subsystems and use either a particular form of the system, or decompose and solve an equivalent system. An overview of these methods have been given by Dragan in [3]. Also, in [3] Dragan proposes an algorithm for decomposing a sparse nonlinear system into subsystems. In [2] Dragan proposes an algorithm for parallel solving of such a system using threads.

2. The Problem

The problem of solving a sparse nonlinear system of equations in parallel can be formulated as follows:

Given the sparse large nonlinear system

$$F(x) = 0,$$

where $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$, solve it in parallel by decomposing the system into subsystems.

The algorithm in [2] uses threads in Win32. Threads can be used in many operating systems. The paper compares threads' usage in Linux and Win32[1], using the same numerical method, Newton SOR, in order to illustrate threads behavior. Because of different tasks scheduling, the same program with threads gives different results on Linux and Win32.

3. Experimental results

Let us consider the system

$$\begin{aligned}
&2x_1x_2 - 2\cos(x_1 - 1) - 10 && = 0\\
&\sin(x_1 + x_3) + \tfrac{x_2}{2} + \sin(x_1 + x_2) && = 0\\
&x_4 \tan(x_3 + x_4) + 12 && = 0\\
&2x_4 - 5x_3 + 2 && = 0\\
&2(x_5 - x_6) - 2\cos(x_7) - \sin(x_8 - 1) && = 0\\
&\tfrac{x_6}{2}\sin(x_5 - x_6) + \sin(x_8 - \sin(x_7)) - 2 && = 0\\
&x_7 - \tan(x_6) + x_5 - x_8 + 5 && = 0\\
&\tan(x_1x_5) + x_6\sin(x_5 + x_8) + 2 && = 0
\end{aligned}$$

If we apply the decomposing algorithm ([3]), the system can be decomposed into three subsystems: $\{1, 2, 3, 4\}$, $\{4, 5\}$ and $\{5, 6, 7, 8\}$.

In the tables below we use the following notations:

[1]By Win32 we understand one of the following operating systems: Windows NT, Windows 2000 or Windows XP.

- x^0 - initial iteration;
- ω - relaxation factor;
- ε - approximation error.

For tests we used:

- an Intel P III/850MHz/512MB RAM to run RedHat Linux 7.2;
- an Intel P IV/3.06GHz/1GB RAM with hyper threading technology to run Windows XP.

Depending on the initial iteration, we consider three cases. For each case we have two tables:

- First table contains number of iterations;
- Second table contains time in milliseconds.

The *none* notation in the tables means that the solution could not be found in $20,000$ iterations.

3.1. $x^0 = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

ω	Win32		Linux	
	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$
0.4	372	612	236	395
0.5	175	274	105	203
0.6	120	168	93	207
0.7	191	165	100	135
0.8	*none*	*none*	*none*	*none*
0.9	60	73	319	357
1.0	177	294	152	269
1.1	393	442	*none*	563

ω	Win32		Linux	
	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$
0.4	13.874	23.010	12.204	20.425
0.5	6.616	10.320	5.466	10.525
0.6	4.592	6.375	4.820	10.722
0.7	7.578	6.209	5.191	6.997
0.8	*none*	*none*	*none*	*none*
0.9	2.313	2.790	16.523	23.527
1.0	6.617	11.630	7.892	13.916
1.1	14.572	16.475	*none*	26.293

In this case one can see that for $\omega \leqslant 0.7$ in Linux we need less iterations, with one exception, for $\omega = 0.6$ and $\varepsilon = 10^{-8}$, when the number of iterations in Linux is greater than in Win32.

Another interesting situation is, for example, for $\omega = 0.6$ and $\varepsilon = 10^{-5}$, when the number of iterations in Win32 is greater than in Linux, but the execution time is shorter. The same situation is for $\omega = 1.0$ and for $\omega = 0.7$ and $\varepsilon = 10^{-8}$.

One can also see that for $\omega = 1.1$ and $\varepsilon = 10^{-5}$ the program cannot be solved in Linux.

3.2. $x^0 = \{4, 4, 4, 4, 4, 4, 4, 4\}$.

ω	Win32		Linux	
	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$
0.4	394	612	211	425
0.5	174	274	123	210
0.6	109	168	65	223
0.7	96	163	100	125
0.8	*none*	*none*	*none*	*none*
0.9	53	70	46	68
1.0	1306	*none*	3345	1056
1.1	156	185	635	678

ω	Win32		Linux	
	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$
0.4	14.695	23.504	8.806	21.974
0.5	6.569	10.371	6.390	10.878
0.6	4.158	6.131	3.407	11.550
0.7	3.651	16.090	8.513	10.674
0.8	*none*	*none*	*none*	*none*
0.9	2.054	2.677	2.428	3.550
1.0	103.455	*none*	176.655	54.611
1.1	3.795	8.047	32.722	34.965

In this case one can easy see some interesting things. For example, for $\omega = 0.7$ and $\varepsilon = 10^{-5}$ the Win32 program has only 4 iterations less than the Linux program, but is 2.331 times faster. For $\omega = 1.0$ the Win32 program cannot solve the system for $\varepsilon = 10^{-8}$ and the Linux program is 3.23 times faster than for $\varepsilon = 10^{-5}$.

3.3. $x^0 = \{8, 7, 6, 5, 4, 3, 2, 1\}$.

ω	Win32		Linux	
	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$
0.4	366	601	183	401
0.5	161	257	116	218
0.6	108	158	60	135
0.7	318	290	60	88
0.8	*none*	*none*	*none*	*none*
0.9	144	109	144	109
1.0	2080	546	879	2326
1.1	174	*none*	17815	482

ω	Win32		Linux	
	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-8}$
0.4	13.872	22.987	9.496	20.722
0.5	6.142	9.686	6.028	11.290
0.6	4.115	5.938	3.810	7.013
0.7	12.103	10.981	3.147	4.579
0.8	*none*	*none*	*none*	*none*
0.9	5.502	4.177	5.282	6.014
1.0	78.319	20.534	45.444	120.182
1.1	6.578	*none*	1699.026	24.865

In the last case we have some other interesting things. For example, for $\omega = 0.9$ we have the same number of iterations in Win32 and Linux. While the Win32 program is faster for $\varepsilon = 10^{-5}$, the situation is changed for $\varepsilon = 10^{-8}$.

Conclusions

One can easy see some differences between the results above and the results obtained in [2]. The differences appear because of hyper threading technology and different task scheduling in Win32. Also for similar number of iterations, the processor speed is very important, even for more iterations.

Under Linux, the system can be solved faster than in Win32 for $\omega \leqslant 0.7$. When the system cannot be solved in Win32, it can be solved in Linux, and, even more, in that case, solving the system in Linux is faster for $\varepsilon = 10^{-8}$ than for $\varepsilon = 10^{-5}$.

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Evolutionary multimodal optimization

D. Dumitrescu, Rodica I. Lung

Abstract. A new evolutionary model for multimodal optimization called the Roaming Genetic Algorithm is proposed. The problem of finding multiple optima is solved using a multi-population model. An external population (an archive) is used to store the optima already found. A stability measure for a sub-population is introduced. Stability is defined using the number of offspring that are better then the best individual in the parent sub-population. Each sub-population evolves in isolation until it becomes unstable. Members of unstable sub-populations are spread over different regions of the search space. This feature is important for maintaining the population diversity. Experimental results prove that the proposed algorithm is able to find multiple solutions even for test functions considered to be deceptive.

1. Introduction

Over the years, genetic algorithms (GA) have proven effective in solving a variety of search and optimization problems. Determining the global optima within a fitness landscape has been the subject of much research. The intrinsic parallelism in a GA suggests, however, that this method should be able to locate several optima of a multimodal function. The problem of locating multiple solutions raises in many real world applications where the knowledge of several different potential solutions provides the decision maker with a better insight into the nature of the design space and perhaps suggest alternative solutions.

Most evolutionary multimodal optimization models are using one way or the other *Fitness sharing* (Goldberg, Richardson, 1987) as the main mechanism for detecting multiple optima. Other approaches are using parallel subpopulations evolving in isolation or in comunication with each other in order to locate the optima. Some of the most popular parallel models are the island models (Gordon et. Al 1992).

The algorithm presented here uses parallel subpopulations evolving in isolation. The subpopulations are 'roaming' in the search space looking for an optimum. At each generation they are verifying whether the next generation will bring them a solution better than the best they already have. If not, they believe that an optimum has been found and will roam in other regions of the search space in order to find other solutions.

2. The Roaming Genetic Algorithm (RGA)

In this section we propose an evolutionary model that identifies the local optima solutions and stores them in a separate population (an archive). The algorithm maintains several subpopulations evolving isolated. The number of subpopulations is a parameter of the algorithm and it is not related to the number of local optima searched. This conffers robustness to the search mechanism. Once a local optimum is detected it is stored in the archive.

Let N be the number of subpopulations. At each generation t the population $P(t)$ consists of N subpopulations $P_i(t)$, $i = 1, \ldots, N$.

We may define an order relation on $P(t)$. Consider a maximization problem. We say that the individual x is better than y and we write

$$x \succ y \quad \Longleftrightarrow \quad eval(x) \geq eval(y).$$

Let x_i^* be the best individual in the subpopulation $P_i(t)$. By evolving subpopulation $P_i(t)$ a new subpopulation $P_i^{'}(t)$ having the same size as $P_i(t)$ is obtained. We define the B operator as the number of offspring in $P_i^{'}(t)$ that are better than x_i^*:

$$B : P(t) \longrightarrow \mathcal{P}(P(t)) \quad B(x_i^*) = \{x \in P_i^{'}(t) \mid x \geq x_i^*\} \tag{1}$$

A stability measure $SM_i(t)$ is associated to each subpopulation $P_i(t)$.

Definition 1. *The stability measure $SM_i(t)$ of the subpopulation $P_i(t)$ is defined*

$$SM_i(t) = \frac{card\, B(x_i^*)}{card P_i(t)},$$

where x_i^* is the best individual in $P_i(t)$ and card A represents cardinality of the set A.

Observation 1. *The stability measure of a subpopulation has several properties that can be deduced from its definition:*

(i) $0 \leq SM_i(t) \leq 1$*;*

(ii) if $SM_i(t) = 0$ *then* x_i^* *is a potential local optimum;*

(iii) $SM_i(t) = 1$ *if all the offsprings in* $P_i^{'}(t)$ *are better than* x_i^* *which means that* $P_i(t)$ *is not near convergence;*

(iv) the operator $IM_i(t)$ *defined for each* $P_i(t)$ *as* $IM_i(t) = 1 - SM_i(t)$ *will be called the* instability measure.

A subpopulation having a stability measure closer to 1 is considered to be *stable*. A stable subpopulation evolves in isolation until detects a local optimum. A subpopulation closer to convergence has a stability measure closer to 0 and it is considered to be *unstable*. A low stablity measure can also indicate that the subpopulation has stucked in a false optimum.

2.1. **Adding a solution to the archive.** An external population called the *archive* is used to store the potential optima.

If a subpopulation $P_i(t)$, $i = 1, \ldots, N$ has $SM_i(t) = 0$ it means that after one iteration no offspring better than x_i^* was produced and we consider x_i^* as a potential local optimum.

A potential optimum can be a local optimum or can be very close to a local optimum.

Before adding a solution x^* to the archive the distance between x^* and every solution in the archive is compared with an archive parameter δ. If this distance is lower than δ, then only the best fitted between x^* and x is saved in the archive. δ is a parameter of the algorithm. Experimental results show that δ depends on the fitness landscape, i.e. the best results are obtained when the value of δ is chosen considering the distance between the local optima of the function.

In conclusion, a solution x^* can be added to the archive only in two situations: either the distance to other solutions in the archive is bigger than δ, meaning that x^* represents a new optimum for the archive, or it is better than other solutions in the archive that are within a distance smaller than δ to x^*.

2.2. **Subpopulation migration.** The stable subpopulations are the subpopulations that have converged or that are near convergence. After adding the potential optima x_i^*, $i \in \{1, \ldots, N\}$ to the archive, the search performed by the subpopulation P_i has to be redirected to other regions of the space. Thus, the stable subpopulations will be spread in the search space in order to locate new optima solutions.

In order to prevent convergence to false optima points *all* the subpopulation are going through a selection process that will choose the ones that have to be redirected. This selection is performed using the instability measure of the subpopulations as an evaluation function. Any type of selection operator (proportional selection, tournament selection etc.) can be used in order to select the unstable subpopulations.

An example of selection operator for subpopulation is presented here. For each subpopulation $P_i(t)$ a random number $q \in [0, 1]$ is generated. Then if q is smaller than IM_i (the unstable subpopulations have better chances to be selected) the subpopulation $P_i(t)$ is selected into a subpopulation migrating pool (SMP).

The subpopulations selected in the (SMP) will be spread in the search space in order to detect other optima. We can say that the subpopulations are migrating to different regions. This migration is realized using genetic operators defined for subpopulations. For example, a mutation operator for subpopulations will apply mutation to each individual of the subpopulation. To ensure complete change of the individuals of the subpopulations, the mutation rate here will be closer to 1.

The following generation $P(t+1)$ will be consist of the subpopulations that were changed in SMP and the offspring $P_i^{'}(t)$ of those subpopulations $P_i(t)$ that were not selected in the SMP.

The algorithm stops after a given number of generations. At the end the archive will contain the local optima solutions.

2.3. The Algorithm.

Input: N - subpopulations number
$Popsize$ - subpopulations size
$Nrgen$ - maximum number of generations
δ - archive parameter
p_c, p_m -crossover probability and mutation rate

Output: $Archive$ - the set containing the local optima

Step 1: *Initialization* a) $t := 0$;
b) Initialize $P_i(0)$ for each $i = 1, \ldots N$ by generating $popsize$ number of individuals;
c) $Archive = \emptyset$;

Step 2: Evaluate each individual x in each subpopulation $P_i(t)$ by caculating its fitness value $F(x)$;

Step 3: Evolve each subpopulation $P_i(t)$ one generation by applying selection, recombination (with the crossover probability p_c) and mutation (with the mutation rate p_m). Let $P_i^{'}(t)$ be the resulting offspring subpopulation.

Step 4: Evaluate each individual x in the offspring subpopulation $P_i^{'}(t)$ by caculating its fitness value $F'(x)$;

Step 5: For each subpopulation $P_i(t)$ calculate:
a) The best individual x_i^*;
b) The stability measure $SM_i(t)$ using Definition 1;

Step 6: For each subpopulation $P_i(t)$ having $SM_i(t)$=0 try to add x_i^* to the $Archive$.

Step 7: For each $i = 1, \ldots N$ do
a) generate a real number $q \in [0, 1]$;
b) if $q \leq IM_i(t)$ then add $P_i(t)$ to SMP;

Step 8: Migrate the subpopulaitons that are in SMP using mutation;

Step 9: Set $P(t+1) = SMP \cup \{P_i'(t) \mid P_i(t) \notin SMP\}$; $t = t + 1$.
If $t < Nrgen$ then go to step 2, else stop.

3. Experimental results

The Roaming Genetic Algorithm has been tested on several standard test functions. In this paper we present the results from tests performed on function $F1$ as defined by Goldberg (1989). Function $F1$ has five peaks of decreasing height in the range $0 \leq x \leq 1$ and it is defined by:

$$F1(x) = e^{-2\ln(2)\left(\frac{x-0.1}{0.8}\right)^2} \sin^6(5\pi x)$$

The exact positions and values of the maxima as given in [1] are shown in Table1. The results presented

Peak	x	Fitness
1	0.1	1
2	0.29942	0.917236
3	0.49883	0.707822
4	0.69825	0.459546
5	0.89767	.0251013

Table 1. F1 peaks

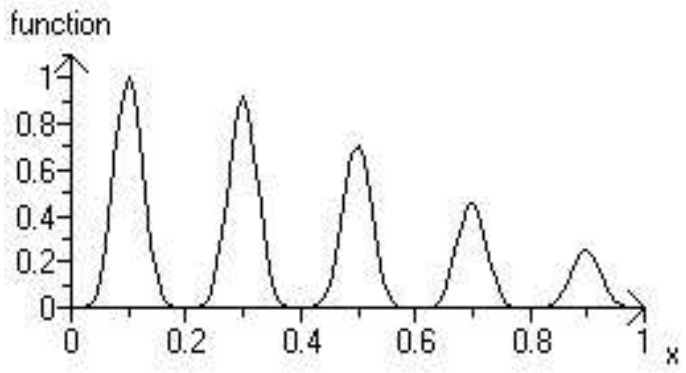

FIGURE 1. Decreasing maxima function F1.

here were obtained using the parameters shown in Table 2.

Subpopulation number	30
Subpopulation size	10
Generation number	150
Parameter δ	0.1
Search precision	6

TABLE 2. Parameters used for the $F1$ function.

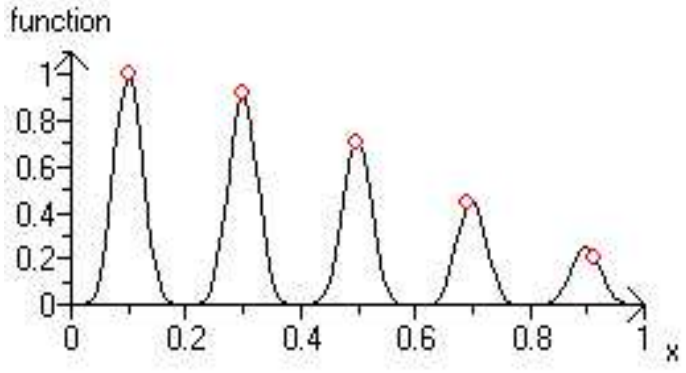

FIGURE 2. The *Archive* after 10 generations.

The algorithm detects the peaks of the function at an early stage as we can see in fig. 2. After 10 generations the algorithm has already detected the peaks, but the solutions found are only close to the optima. The number of generations needed to refine these solutions depends on the robustness of the genetic operators used in Step 2. For this example we used binary tournament selection, a 2-point crossover operator with the crossover probability 0.5 and strong mutation with the mutation rate 0.05.

The number of generations needed to refine the solutions depends also on the number of subpopulation used. The reason is obvious: more populations have more chances to locate the optima.

In Table 3 the solutions averaged over 10 RGA runs are presented.

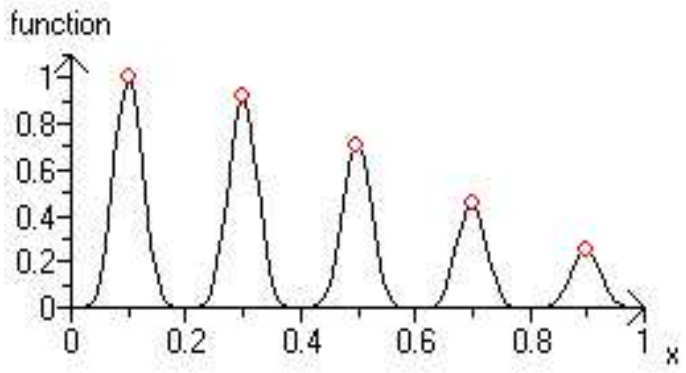

FIGURE 3. The *Archive* after 150 generations

Peak	x	Fitness
1	0.100000	1.000000
2	0.299350	0.917223
3	0.498810	0.707820
4	0.698570	0.459510
5	0.897740	0.251010

TABLE 3. Numerical results after 150 generations.

4. CONCLUSIONS

A new evolutionary model for multimodal optimization called the Roaming Genetic Algorithm (RGA) is presented. RGA uses a number of roaming subpopulations in order to detect multiple optima. A measure for the stability of a sub-population is introduced in order to asses wether a subpopulation has located an optimum or not. The subpopulations evolve in isolation until they detect an optimum. The optima detected are saved into an archive and the corresponding subpopulations are spread to other regions of the search space. An example is presented to prove the efficiency of the algorithm. The RGA is capable to locate the peaks of the function but it needs improvments as far as the accuracy of the solutions is concerned. A solution for this problem might be the use of local search in order to improve the individuals in the archive.

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Double product MV-algberas versus product MV-algebras

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Abstract. Product MV-algebras and Double Product MV-algebras are generalizations of MV-algebras looking very similar. Our aim is to investigate the relationship between these algebraic structures.

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1. Introduction

MV-algebras were introduced by Chang in 1958 (see [1]) and since then this structure has captured the interest of many mathematicians. An equivalent definition of MV-algebras is given by Mundici in ([2]), as follows:

Definition 1. *An MV-algebra is an algebra $(M, \oplus, \neg, 0_M)$ with a binary operation $\oplus$, a unary operation $\neg$ and a constant 0_M satisfying the following equations:*
(i) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(ii) $x \oplus y = y \oplus x$;
(iii) $x \oplus 0_M = x$;
(iv) $\neg\neg x = x$;
(v) $x \oplus \neg 0_M = \neg 0_M$;
(vi) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Remark 2. *The constant 1_M and the operations $\odot$ and $\ominus$ are defined on each MV-algebra M as it follows:*
(i) $1_M = \neg 0_M$;
(ii) $x \odot y = \neg(\neg x \oplus \neg y)$;
(iii) $x \ominus y = x \odot \neg y$.

On this structure there is defined a partial order relation as follows:

Definition 3. *Let M be an MV-algebra and $x, y \in M$. We say that $x \leq y$ if and only if x and y satisfy one of the bellow equivalent conditions:*
(i) $\neg x \oplus y = 1_M$;
(ii) $x \odot \neg y = 0_M$;
(iii) $y = x \oplus (y \ominus x)$;
(iv) there is an element $z \in M$ such that $x \oplus z = y$.

Mundici ([5]) proved that there exists a categorial equivalence between the category of MV-algebras and unital (abelian) l-groups. It means that any MV-algebra is obtained from an l-group G with strong unit.

In what follows we will describe the way of obtaining an MV-algebra from an l-group G with strong unit.

Let $(G, +, 0, \leq)$ be an abelian l-group. We say that $u \in G$ is *a strong unit* of G if for any $v \in G$ there is an integer $n \geq 1$ such that $-nu \leq v \leq nu$.

Let us consider now the following operations:

(i) $\oplus : [0,u] \times [0,u] \to [0,u]$ where for any $x, y \in [0,u]$ we have $x \oplus y = (x+y) \wedge u$;

(ii) $\neg : [0,u] \to [0,u]$ where for any $x \in [0,u]$ we have $\neg x = u - x$;

then the structure $([0,u], \oplus, \neg, 0)$ is an MV-algebra.

Di Nola and Dvurecenskij ([7]) introduced Product MV-algebras by enriching MV-algebras with a new internal binary operation called product.

Definition 4. *A Product MV-algebra (shortly PMV) is an algebraic structure* $(M, \oplus, \cdot, \neg, 0_M)$ *fulfilling the following axioms:*

(i) $(M, \oplus, \neg, 0_M)$ *is an MV-algebra;*

(ii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

(iii) if $a + b$ *is defined in* M, *then* $a \cdot c + b \cdot c$ *and* $c \cdot a + c \cdot b$ *exists and*

$$(a+b) \cdot c = a \cdot c + b \cdot c,$$

$$c \cdot (a+b) = c \cdot a + c \cdot b,$$

for any $a, b, c \in M$.

It is important to notice that the binary operation $+$ used to define the product MV-algebras is the binary operation from the l-group G that generates the MV-algebra $(M, \oplus, \neg, 0_M)$.

Also they have proved that there exists a categorial equivalence betveen the category of associative l-rings with a strong unit u such that $u \cdot u \leq u$ and the category of Product MV-algebras.

In the same time that Di Nola and Dvurecenskij introduced product MV-algebras, Dumitrescu ([3], [4]) introduced Double Product MV-algebras also by enriching the MV-algebra with a binary internal operation $\cdot$ called product. The name was chosen Double Product MV-algebras because the operation $\oplus$ induces in an MV-algebra the product $\odot$, meaning that the new product is the second one.

The definition of this new structure is very similar to the definition of Product MV-algebras, but as we will see later it is not equivalent.

Definition 5. *A Double Product MV-algebra (shortly DMV) is an algebraic structure* $(M, \oplus, \cdot, \neg, 0_M)$ *fulfilling the following axioms:*

(i) $(M, \oplus, \neg, 0_M)$ *is an MV-algebra;*

(ii) $(M, \cdot)$ *is a semigroup;*

(iii) if $a \odot b = 0_M$ *then* $c \cdot (a \oplus b) = c \cdot a \oplus c \cdot b$ *and* $(a \oplus b) \cdot c = a \cdot c \oplus b \cdot c$ *for any* $a, b, c \in M$.

As we can see the new product $\cdot$ is considered to be distributive according to the operation $\oplus$ defined in the MV-algebra M not according to the operation $+$ defined in the l-group that generates the MV-algebra M.

Remark 6. *In what follows we will denote the class of PMV by* C_{PMV} *and the class of DMV by* C_{DMV}.

2. PMV versus DMV

Since the definitions of PMV and DMV look very similar, we intend to determine that these structures are not equivalent. We also will determine which one is more general and will give an example that will sustain our result.

The PMV and the DMV have the same structure, which is $(M, \oplus, \cdot, \neg, 0_M)$ but the axioms are not all the same.

Both these structures start from the MV-algebra $M = (M, \oplus, \neg, 0_M)$.

In the definition of PMV the second axiom is equivalent with the fact that $(M, \cdot)$ is a subgroup which is the second axiom in the definition of DMV.

It follows that if these two structures are not equivalent it is because of the axioms regarding the distributivity of the product $\cdot$.

Let us compare now the two axioms. In the axiom (iii) of the definition of PMV we have the following condition:

if $a + b$ is defined

which is equivalent to

$a + b \leq 1_M$.

This relation is also equivalent with

(1) $a \leq 1_M - b = \neg b$.

¿From (ii) definition (11) and relation (12) we obtain:

$a \odot b = 0_M$

which is the condition of axiom (iii) from definition of DMV.

It means that the two conditions are equivalent and if the two structures are not equivalent it is only because of the definitions of distributivity.

We have now only to compare the distributivity conditions that appear in the both definitions of the new structures PMV and DMV.

In the definition of PMV we have the distributivity introduced by using the binary operation + from the l-group that generates the MV-algebra M as it follows:

(2) $(a + b) \cdot c = a \cdot c + b \cdot c$.

Since $a + b$ and $a \cdot c + b \cdot c$ exist (see (iii) Definition (3)) it means that $a + b \leq 1_M$ and $a \cdot c + b \cdot c \leq 1_M$. It follows that

(3) $a \oplus b = (a + b) \wedge 1_M = a + b$

and

(4) $a \cdot c \oplus b \cdot c = (a \cdot c + b \cdot c) \wedge 1_M = a \cdot c + b \cdot c$.

From equations (2), (3) and (4) we obtain

$(a \oplus b) \cdot c = a \cdot c \oplus b \cdot c$

which is the definition of distributivity from the definition of DMV (see (iii) Definition (4)).

In a similar way we will obtain that if

$c \cdot (a + b) = c \cdot a + c \cdot b$

in the conditions of definition (3) it follows that

$c \cdot (a \oplus b) = c \cdot a \oplus c \cdot b$.

The above results lead us to the folowing theorem:

Theorem 7. *Any Product MV-algebra is an Double Product MV-algebra.*

Proof. The proof of this theorem is sustained by the results obtained above. □

Remark 8. *The above theorem means that if* $P \in C_{PMV} \Rightarrow P \in C_{DMV}$.

In what follows we will check if the reverse of theorem (7) also holds true. To reach a conclusion in what concerns us, we will start from the definition of distributivity from definition (4):

(5) $(a \oplus b) \cdot c = a \cdot c \oplus b \cdot c$.

Since $a \odot b = 0_M$ (see (iii) definition (4)) it follows that

(6) $a \oplus b = (a + b) \wedge 1_M = a + b$.

¿From equations (5) and (6) we have

$(a + b) \cdot c = a \cdot c \oplus b \cdot c = (a \cdot c + b \cdot c) \wedge 1_M$.

It follows that if $a \cdot c + b \cdot c \leq 1_M$ we have

(7) $\quad (a+b) \cdot c = a \cdot c + b \cdot c$

which is the definition of distributivity from definition (3), but it also follows that if $a \cdot c + b \cdot c > 1_M$ the equation (7) is not true.

Since we can not know that always $a \cdot c + b \cdot c \leq 1_M$, the above results lead us to the following theorem:

Theorem 9. *Not any Double Product MV-algebra is an Product MV-algebra.*

Remark 10. *The above theorem means that even if $D \in C_{DMV}$, there are situations when $D \notin C_{PMV}$.*

The theorems (7) and (9) lead us to the main result of this paper:

Theorem 11. *The class of Product MV-algebras is strictly included in the class of Double Product MV-algebras.*

Proof. From remark (8) we have that

(8) $\quad C_{PMV} \subseteq C_{DMV}.$

From remark (10) we also have that

(9) $\quad C_{DMV} \nsubseteq C_{PMV}.$

From relations (8) and (9) we have that

$$C_{PMV} \subset C_{DMV}$$

that completes the proof. □

Remark 12. *Since $C_{PMV} \subset C_{DMV}$ it follows that Double Product MV-algebras is a more general structure than Product MV-algebras.*

In the next section we will present an example that proves that the inclusion in theorem (11) is strict.

3. An example of structure that is an DMV and is not an PMV

In ([6]) it was proved that the structure $([0, 2^t - 1], \oplus, \neg, 0)$ is an MV-algebra with the operations defined as follows:

$$a \oplus b = (a + b) \wedge \left(2^t - 1\right)$$

and

$$\neg a = 2^t - 1 - a.$$

Let $t = 2$. It follows that $([0, 3], \oplus, \neg, 0)$ is an MV-algebra with the operations defined as follows:

$$a \oplus b = (a + b) \wedge 3$$

and

$$\neg a = 3 - a.$$

Let us now consider the binary multiplicative operation $\bullet : [0, 3] \times [0, 3] \to [0, 3]$ defined as follows:

$$a \bullet b = (a \cdot b) \wedge 3,$$

where the binary operation $\cdot$ is the usual product of real numbers.

It is easy to prove that $([0, 3], \bullet)$ is a subgroup and that the structure $([0, 3], \oplus, \bullet, \neg, 0)$ is a Double Product MV-algebra.

Let $x = 1.4$, $y = 1.4$ and $z = 1.2$ be three numbers from $[0, 3]$ interval. It is obvious that $x + y \leq 3$.

Let us assume that $([0, 3], \oplus, \bullet, \neg, 0)$ is a Product MV-algebra. Since $x + y \leq 3$ it follows that

(10) $\quad (x + y) \bullet z = x \bullet z + y \bullet z.$

But

$$\begin{aligned}(x+y)\bullet z &= (1.4+1.4)\bullet 1.2\\ &= 2.8\bullet 1.2\\ &= 3.36\wedge 3\\ &= 3\end{aligned}$$

and

$$\begin{aligned}x\bullet z+y\bullet z &= 1.4\bullet 1.2+1.4\bullet 1.2\\ &= 1.68\wedge 3+1.68\wedge 3\\ &= 1.68+1.68\\ &= 3.36.\end{aligned}$$

Since $3.36\notin[0,3]$ it follows that $x\bullet z+y\bullet z$ is not defined in $[0,3]$ and it follows that equation (10) does not hold for these values of x, y and z. It means that the assumptiom that $([0,3],\oplus,\bullet,\neg,0)$ is a Product MV-algebra is not correct.

It follows that even if the structure $([0,3],\oplus,\bullet,\neg,0)$ is a Double product MV-algebra, it is not a Product MV-algebra.

This example shows that the inclusion in the theorem (11) is strict.

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Reducing complexity of RBF neural networks by dynamic evolutionary clustering techniques

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Abstract. Radial Basis Function (RBF) is a very popular neural network model. RBF networks are used in many domains, especially for solving interpolation problems. Complexity of RBF networks depends on the number of the processor units in the hidden layer. This number can be reduced by using some techniques for clustering training data. The most popular clustering method for solving this problem is the K-Means algorithm. K-Means is a static method. It requires prior specification of the number of clusters. A new dynamic evolutionary technique, called Genetic Chromodynamics (GC), is used for detecting the optimal cluster number and the corresponding classes. Some experiments using GC are described.

1. Introduction

Radial Basis Function (RBF)([8],[1],[5],[7]) is a relatively simple neural network with two active layers. The activation functions for the processor units of the hidden layer are radial basis functions (for example Gauss functions). These functions generally have two parameters: the center and the width.

The complexity of the network (and also the speed of the training process) depends on the number of the radial basis functions with different center parameters. The correct determination of the center number and position is a primary problem. If few training points are present, then we can use all of them as center parameters for the radial functions. In this case the number of the processor units in the hidden layer is equal to the number of the training points. If the number of the training points is high, we can not use all of them. In this situation a single neuron for a group of similar training points can be considered.

These groups of similar training points can be identified by using clustering methods. By clustering a data set is divided into regions of high similarity, as defined by some distance (metric). In most instances, a prototypical vector (the cluster center) identifies a cluster. Hence, the problem of cluster optimization is twofold: optimization of cluster centers and optimization of number of clusters. The latter aspect has often been neglected in standard approaches (static clustering methods), as these typically fix the number of clusters a priori. In contrast to static, dynamic clustering does not require the priori specification of the number of clusters.

The most popular method for the determination of the center parameters for the RBF neural networks is the K-Means algorithm [10]. K-Means has many disadvantages. An improved version of K-Means is the Generalized K-Means algorithm [9]. Generalized K-Means also has some drawbacks. The most important disadvantage is that it requires prior specification of the number of clusters.

To overcome this shortcoming of fixing cluster number *a priori*, there were some tentative to develop dynamic evolutionary clustering algorithms.

A dynamic evolutionary method could be more efficient for the determination of the center parameters for the RBF neural networks. Some experiments using a new dynamic evolutionary clustering method, the Genetic Chromodynamics (GC) ([2],[3],[6]), are described. The GC method is described in the next section and after that some results are presented. A standard interpolation problem is considered, and is solved by using RBF neural network. To reduce the network complexity clustering methods are used. Both Generalized K-Means and GC are used, and the obtained results are compared.

2. Genetic chromodynamics

GC is a new kind of evolutionary search and optimization metaheuristics ([2],[3]). GC is a metaheuristics for maintaining population diversity and for detecting multiple optima. The main idea of the strategy is to force the formation and maintenance of stable sub-populations.

GC based methods use a variable-sized population, a stepping-stone search mechanism, a local interaction principle and a new operator for merging very close individuals.

Corresponding to the stepping-stone technique each individual in the population has the possibility to contribute to the next generation and thus to the search progress. The stepping-stone search combines the advantages of elitist and diversity maintaining strategies.

Local interaction principle may have several expressions. For instance, we may consider the local mating and the local subpopulation recombination models. Recombination mate of a given individual is selected within a determined mating region. Only short range interactions between solutions are allowed. Local mate selection is done according to the values of the fitness function. An adaptation mechanism can be used to control the interaction range, so as to support sub-population stabilization. Within this adaptation mechanism the interaction radius of each individual could be different.

To enhance GC, micropopulation models can be used [6]. For each individual we may consider a local interaction domain. Individuals within this domain represent a micropopulation. All solutions from a micropopulation are recombined using local tournament selection. If the local domain of an individual is empty, then the individual is mutated.

Within GC sub-populations co-evolve and eventually converge towards several optima. The number of individuals in the current population usually changes with the generation. Very close individuals are merged. A merging operator is used for merging very close individuals. At convergence the number of sub-populations equals the number of optima. Each final sub-population hopefully contains a single individual representing an optima, a solution of the problem.

GC allows any data structure suitable for the problem together with any set of meaningful variation/search operators. For instance, solutions may be represented as real-component vectors. Moreover, the proposed approach is independent of the solution representation.

In the following a GC based clustering method is presented. This method is used for solving our problem.

2.1. **Solution representation.** Let

$$X = \{x_1, ..., x_m\}, x_i \in \mathbf{R}^s, s \geq 1,$$

be the data set for clustering. The cluster structure of X is given by a fuzzy partition $P = \{A_1, ..., A_n\}$ of X. Every class A_i is represented by a prototype $L_i \in \mathbf{R}^s$. $L = \{L_1, ..., L_n\}$ is the representation of the partition P.

In the proposed clustering technique each prototype is encoded into a chromosome. Totality of these chromosomes represents a generation.

The idea of the method is to determine formations of evolving chromosomes converging towards prototypes of real clusters.

The initial population is randomly generated and it contains a large number of individuals. The operations involved in the searching process are selection, crossover, mutation and merging.

2.2. **Interaction range.** For each individual in the population (a chromosome representing a prototype) a mating region is considered as the closed ball with center c and radius d^*

$$V(c, d^*) = \{y | d(c, y) \leq d^*\},$$

where the *interaction radius* d^* depends on the chromosome.

Initially we consider the neighborhood distance for each chromosome as the standard deviation of all points. For a chromosome L the mean distance $\overline{\delta}$ between the points in $V(L, d^*)$ and L is given by

$$\overline{\delta} = \sum_{i=1}^{n_{d^*}} \frac{d(x_i, L)}{n_{d^*}},$$

where $x_1, ..., x_{n_{d^*}}$ are the points in the neighborhood with radius d^* of L.

When the points in $V(L, d^*)$ are uniformly distributed, the value of $\overline{\delta}$ is $\frac{d^*}{\beta}$, where $\beta \in (1, 2]$ is a fixed number, which depends on the dimension s of the search space (generally the best value for β is $\sqrt[s]{2}$).We want to adjust d^* such that $\overline{\delta}$ to be equal to $\frac{d^*}{\beta}$, Thus if $\overline{\delta} \leq \frac{d^*}{\beta}$, then the next value for d^* is $\beta\overline{\delta}$, else $\overline{\delta}$. If there are not at least two points in the neighborhood of the chromosome, then the previous distance value is not modified.

2.3. **Fitness function.** The considered fitness function is a sum of radial basis functions centered at the points of the set that has to be classified.

Fitness value of the chromosome L may be written as

$$f(L) = \sum_{i=1}^{m} \frac{1}{d^{\alpha}(x_i, L) + C},$$

where $\alpha \geq 1$ and $C > 0$.

The role of the constant C is to prevent infinite or too great values for the fitness function, and together with α controls the granularity of the clusters.

2.4. **Selection.** A micropopulation model and a local tournament selection are used.

At each step of the generation process every chromosome is selected to produce an offspring through crossover or mutation. The mate for the crossover operation for an individual s is selected among the chromosomes in its neighborhood with a proportional selection. In this case the selection probability of an individual c_i in the neighborhood of s is

$$p_i = \frac{f(c_i)}{\sum_{c_j \in V(s, d^*)} f(c_j)}.$$

A random number $r \in [0, 1]$ is generated, if the condition

$$\sum_{i=1}^{j-1} p_i < r \leq \sum_{i=1}^{j} p_i$$

is fulfilled, then the chromosome c_j will be selected as a mate of s.

Later the mate will be selected as first parent to produce its offspring, for this reason at crossover only one new chromosome is generated. If there is no mate for the crossover operation in the neighborhood of radius d^* of an individual, then the mutation operator is applied.

2.5. **Crossover.** An individual can be involved into a crossover operation only with individuals that are at smaller distance than d^*. The crossover operation is a convex combination of the codes of the genes. The coefficient of the combination is a randomly generated number for each gene.

Consider the chromosome $c = (c_1, ..., c_s)$ and the mate $d = (d_1, ..., d_s)$ selected for crossover. Offspring chromosome is $o = (o_1, ..., o_s)$:

$$o_i = \alpha_i c_i + (1 - \alpha_i) d_i, i = 1, ..., s,$$

where α_i are random coefficients having uniform distribution in $[0, 1]$.

2.6. **Mutation.** Mutation is an additive perturbation of the genes with a randomly chosen value from a normal distribution $N(0, \sigma)$. Offspring of chromosome $c = (c_1, ..., c_s)$ is $o = (o_1, ..., o_s)$, obtained as

$$o_i = c_i + r_i,$$

where r_i has the normal distribution $N(0, \sigma)$. Therefore we may write

$$r_i = \sigma N_i(0, 1),$$

where σ is the *mutation step size* and $N_i(0, 1)$ is the realization of a normal random variable.

2.7. **Survival.** At each generation every chromosome is involved in crossover or mutation. An offspring can replace only its parent. When an offspring is produced, it is compared with the parent and the best (with better fitness) is introduced in the new generation.

2.8. **Merging.** An effect of the crossover operation is that the chromosomes in the same subpopulation are overlapping after a number of iterations. When the distance between two chromosomes is smaller than a considered value ε (*merging radius*), they are merged. In this way, the size of the population decreases during the process until n individuals are remained, where n is the optimal cluster number.

2.9. **Termination and fuzzy class detection.** If after a fixed number of iterations there are not changes in the population, the process stops and the individuals constituting the last population are considered as prototypes of the detected clusters. For all points the fuzzy membership to the clusters determined by the prototypes is calculated according to the formula

$$A_i(x_j) = \frac{1}{\sum_{k=1}^{n} \frac{d^2(x_j, L_i)}{d^2(x_j, L_k)}},$$

where $i = 1, ..., n; j = 1, ..., m$.

3. Numerical Experiments

We consider a standard interpolation problem.

A RBF neural network is used to approximate the function

$$f : [0,1] \to R, f(x) = \left(x - \frac{1}{3}\right)^3 \cdot \frac{1}{27} + 4y.$$

At the l^{th} step of the learning process the global *learning error* is calculated according to the formula

$$E_l = \frac{1}{N} \sum_{i=1}^{N} (z_i - y_i)^2,$$

where N is the number of the points in the training data set, z_i is the desired output and y_i is the network output.

Consider M inputs that do not belong to the training set. The *generalization error* associated with M is calculated according to the formula

$$E_g = \frac{1}{M} \sum_{i=1}^{M} (z_i - y_i)^2.$$

3.1. **Experimental Condition.** The learning rate for the training process is fixed to 0.1. The learning process stops if the global learning error decreases to 0.00005.
In our experiments the generalization error is calculated using $M = 400$ inputs from the interval $[0, 1]$. Both the Generalized K-Means and the GC algorithms are used for clustering data.

The parameters for the genetic clustering algorithm are:

- initial population size: 200;
- parameters for the fitness function: $\alpha = 1, C = 0.00001$;
- mutation step size: $\sigma = 0.00001$;
- merging radius: $\varepsilon = 0.02$.

The clustering process stops if there is no change in the populations after 50 iterations, or if the number of iterations reaches 5000.

3.2. **Experiment 1.** A fixed input data set, with 100 data points, organized in 18 clusters is considered. The GC algorithm founds the number of the clusters and a center for each of them (fig. 1.). Using these 18 centers for training, in 10945 epoches the 0.00005 global learning error is achieved. The generalization error is 3.442700794496429E-4.

For the K-Means algorithm the number of the centers is randomly generated between 10 and 25 (we suppose that there are more than 10 and less than 25 clusters). 10 tests with 10 different values for the number of the centers were performed. The results are presented in Table 1.

The mean generalization error using the K-Means algorithm is: 0.002228871.

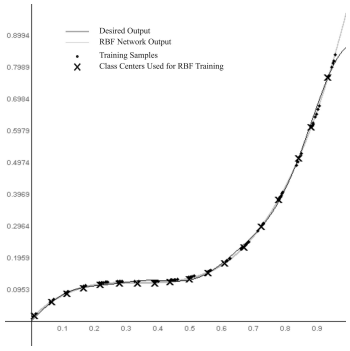

FIGURE 1. Data set with 100 training samples organized in 18 clusters, centers determined by the GC algorithm and the output of the RBF Neural Network after 10945 epoches.

No. of Centers	No. of Epoches	Generalization Error
10	42386	0.003929447755582894
11	26312	0.0039125335843709025
12	15889	0.0038635588999552293
14	8218	0.0037191067346458145
16	2153	0.0028095882895919533
17	2479	0.002400189413222201
18	5466	7.485072155731134E-4
19	10208	5.057298901372404E-4
20	10017	2.240292093213028E-4
23	4918	1.76023288279397E-4

TABLE 1. Generalization errors obtained in 10 runs, using the Generalized K-Means algorithm and 10 different values for the number of centers.

A better result is obtained with the GC based dynamic clustering technique. The generalization error for GC is smaller. GC is able to determine the optimal number of the centers. Using the K-Means method much better result is obtained by using 18 or greater value for the number of the centers, than using 17 or a smaller value (18 was the real number of the centers). The learning process is very sensitive to the number of the clusters.

3.3. **Experiment 2.** 10 data sets with 100 randomly distributed points are considered. Both the K-Means and the GC algorithms are performed. The number of the centers for the K-Means algorithm is fixed to 25. The obtained results are presented in Table 2. and Table 3.

The mean generalization error is 0.003419943 using Generalized K-Means and 0.003357283 using GC.

Better result is obtained by using the GC based method. The mean generalization error is smaller for the GC algorithm. Using GC the number of the cluster centers is not a priori specificated.

3.4. **Conclusions.** In both experiments better results were obtained by using the genetic algorithm. The mean of the generalization errors is smaller for GC technique. Experiment 1 proved that the learning process is very sensitive to the number of the clusters (this is *a priori* specificated in the case of the static clustering method). The GC based dynamic evolutionary clustering method is able to

Generalized K-Means			
Id. set	No. of Centers	No. of Epoches	Generalization Error
1	25	3796	0.004910095167999367
2	25	3403	0.0015323728028574296
3	25	3531	0.0031077660275593572
4	25	3477	0.0011937407917221928
5	25	2007	0.0029516540972555575
6	25	2418	0.003354267453187255
7	25	3508	0.004110996687588495
8	25	3662	0.001079151961924775
9	25	2743	0.006245060418005246
10	25	4119	0.00571432481480872

TABLE 2. Generalization errors obtained in 10 different runs, using the Generalized K-Mean algorithm and 10 data sets with 100 randomly distributed samples.

Genetic Chromodynamics			
Id. set	No. of Centers	No. of Epoches	Generalization Error
1	25	3831	0.0048167188232168335
2	28	1377	0.0019146090647390935
3	26	2510	0.002650884708119015
4	28	1568	8.894281720724966E-4
5	26	1722	0.0027860163173894677
6	28	1494	0.003651758302036659
7	25	3541	0.004255004659509415
8	24	3318	8.108781955866825E-4
9	27	1861	0.006416108804838487
10	26	3097	0.0053814183758560405

TABLE 3. Generalization errors obtained in 10 different runs, using the Genetic Chromodynamics algorithm and 10 data sets with 100 randomly distributed samples.

determine the optimal number of the class centers. These centers are used as training data for our RBF network. Automated detection of training centers is an important advantage, especially in the case when the number of training samples is high. In this situation we can not fix this parameter manually.

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A method of solving multiple objective linear fractional programming problems

Silviu Dumitrescu, Bogdana Pop

Abstract. The aim of this paper is to develop an interactive method of solving multiple objective linear fractional programming problems using an extended Rosen algorithm.
Starting from the idea of Rosen's algorithm, in 1999 Cocan and Pop proposed a method of solving convex programming problems with several objective functions optimizing more than one criteria over a common direction. When the case of linear fractional programs is considered, the optimization of a convex one input value function should be changed by the algebrical selection of the optimal value of a decreased one input value function. The advantage of this change is proved and a new algorithm is formulated here. On the basis of the proposed method illustrative numerical examples are solved.

Keywords: linear fractional programming, multiple objective programming.

1. Introduction

In this paper, an interactive method of solving multiple objective linear fractional programming problems using an extended Rosen algorithm is developed. Extended Rosen Algorithm (ERA) as an extension of the classical algorithm of the convex programming is presented in Section 2 in order to solve a linear fractional programming problems with one objective function. Other algoritms in linear fractional programming problems are presented in [3].

Starting with the idea of the Rosen's algorithm [2], in [1] Cocan and Pop proposed a method of solving convex programming problems with several objective functions optimizing more than one criteria over a common direction. A multiple objective linear fractional programming problem is considered now. In such a case, the optimization input value function should be changed by the algebraical selection of the optimal value of a decreased one input value function. The advantage of this change is proved in Section 3 and a new algorithm is formulated in Section 4. On the basis of the proposed method an illustrative numerical example is solved in Section 5.

2. Extended Rosen Algorithm

Consider the linear fractional programming problem

$$\min\left\{f(x)=\frac{N(x)}{D(x)} \mid x\in X\right\} \tag{1}$$

where

(i) $X=\{x\in R^n \mid Ax\leq b\}$ is a convex and bounded set,
(ii) A is an $m\times n$ constraint matrix, $b\in R^m$, x is an n-dimensional vector of decision variable,
(iii) $N(x)=c^Tx+c_0,\ D(x)=d^Tx+d_0,\ c,d\in R^n, c_0,d_0\in R,\ d^Tx+d_0>0, \forall x\in X$.

Let be

(i) $\widehat{x}$ a feasible solution for problem (1),
(ii) A_r a maximal submatrix of matrix A having the form $(a_{i_1},a_{i_2},...,a_{i_r})^T$, such that $a_{i_j}\widehat{x}=b_{i_j}$, $i_j\in\{1,...,m\}$, $j=1,...,r$;
(iii) $P_r=I_n-A_r^T\left(A_rA_r^T\right)^{-1}A_r$ the projection matrix over the subspace $D=\{x\in R^n \mid a_{i_j}x=0,\ j=1,...,r\}$ (according to [1]);
(iv) $\left(\widehat{d}\right)^T=-P_r\left(\nabla f\left(\widehat{x}\right)\right)^T$ the P_r projection of the gradient of function f;
(v) $\widehat{I}=\left\{\lambda\in R \mid \widehat{x}+\lambda\widehat{d}\in X\right\}$;

(vi) $\varphi : \widehat{I} \to R, \varphi(\lambda) = f\left(\widehat{x} + \lambda\widehat{d}\right)$ the restriction of function f over $\left\{x \in X \mid \exists \lambda \in R,\ x = \widehat{x} + \lambda\widehat{d}\right\}$.

Proposition 1. *P_r is a positive defined matrix.*

Proposition 2. *P_r is a symmetric matrix.*

Proposition 3. *$\widehat{I}$ is an interval $[\alpha, \beta]$ in R, where*

$$\alpha = \max_{i|a^i\widehat{d}<0}\left\{\frac{b_i - a^i\widehat{x}}{a^i\widehat{d}}\right\}, \beta = \min_{i|a^i\widehat{d}>0}\left\{\frac{b_i - a^i\widehat{x}}{a^i\widehat{d}}\right\}.$$

Proposition 4. *Function φ decreases on $\widehat{I}$.*

Thereby, Extended Rosen Algorithm (ERA) to solve problem (1) is:

- Step 1. Read the coefficients of problem (1), c, c_0, d, d_0, A, b as input values.
- Step 2. Find $x^0 \in X$ as a feasible solution of problem (1) which satisfies exactly n equalities and $(m-n)$ strictly inequalities of the system of constraints. Initialize $k = 0$.
- Step 3. (Optimality test) Evaluating

(2) $$P_r\left(\nabla f\left(x^k\right)\right)^T = 0$$

(3) $$\left(A_r A_r^T\right)^{-1} A_r\left(\nabla f\left(x^k\right)\right) \leq 0$$

if both (2) and (3) are true then STOP, with x^k as a solution of Problem (1). Otherwise, choose $\left(d^k\right)^T = -P_r\left(\nabla f\left(x^k\right)\right)^T$ if (2) is false or $\left(d^k\right)^T = -P_{r-1}\left(\nabla f\left(x^k\right)\right)^T$ if (2) is true and (3) is false, as a direction of minimization.

- Step 4. Compute $x^{k+1} = x^k + \lambda_{max} d^k$, where

$$\lambda_{max} = \min_{p|a^p d^k>0}\left\{\frac{b_p - a^p x^k}{a^p d^k}\right\}$$

and go to Step 3 with $k = k + 1$.

3. The model of multiple objective linear fractional programming problem

Consider the multiple objective linear fractional programming problem

(4) $$\text{" min "}\left\{z(x) = \left(\frac{N_1(x)}{D_1(x)}, \frac{N_2(x)}{D_2(x)}, ..., \frac{N_p(x)}{D_p(x)}\right) \mid x \in X\right\}$$

where

(i) $X = \{x \in R^n \mid Ax \leq b\}$ is a convex and bounded set,
(ii) A is an $m \times n$ constraint matrix, x is an n-dimensional vector of decision variable and $b \in R^m$, $p \geq 2$,
(iii) $N_i(x) = \left(c^i\right)^T x + d_i,\ D_i(x) = \left(e^i\right)^T x + f_i, \forall i = \overline{1,p}$,
(iv) $c^i, e^i \in R^n, d_i, f_i \in R, \forall i = \overline{1,p}$,
(v) $\left(e^i\right)^T x + f_i > 0, \forall i = \overline{1,p}, \forall x \in X$.

The term "*min*" being used in problem (4) for finding all efficient solutions in a minimization sense ([3]).

In the following, we assume that a condition for a current point x^* to be accepted as the solution of problem (4) is to satisfy the inequality $\sum_{h=1}^{p}\left(z_h - z_h(x^*)\right)^2 < K$ where $z_h = \min\{z_h(x) \mid x \in X\}$, $h = 1, ..., p$. These facts are formulated according to the definitions described by Cocan and Pop in [1].

Using only the differentiability of the objective functions $f_{i_1}, f_{i_2}, ..., f_{i_q}$, $i_j \in \{1, 2, ..., r\}$, $j = 1, 2, ..., q$, Theorem 3.2 from [1] proved that $s \in R^n$ is a common direction of minimization for the values of the above mentioned criteria in x_0 iff $s(\nabla f_{i_j}(x_0))^T < 0, j = 1, 2, ..., q$.

Proposition 5. *Let $s^* \in R^n$ be a common direction of minimization for the values of criteria $f_{i_1}, f_{i_2}, ..., f_{i_q}$, $i_j \in \{1,2,...,p\}$, $j = 1,2,...,q$, in x^*. The minimal values of $f_{i_1}, f_{i_2}, ..., f_{i_q}$ are obtained at the point $x^* + \lambda^* s^*$, where*

$$\lambda^* = \min_{i|a^i s^* > 0} \left\{ \frac{b_i - a^i x^*}{a^i s^*} \right\}.$$

4. The solution method

- Step 1. Establish the constant value K and put $i = 0$.
- Step 2. Compute values $(z_h)_{h=1,2,...,p}$ using ERA.
- Step 3. Choose $x^0 \in X$ as a feasible solution of problem (4).
- Step 4. If $\nabla z(x^i) = 0$, then x^i is the optimal solution of the problem. Otherwise go to Step 5.
- Step 5. (Acceptability test) Compute $S\left(x^i\right) = \sum_{h=1}^{p} \left(z_h - z_h\left(x^i\right)\right)^2$.
  - If $S\left(x^i\right) < K$, then favorable STOP with x^i the acceptable solution of the problem.
  - If $S\left(x^i\right) \geq K$ and a $h \in \{1, \ldots, r\}$ exists such that $(z_h - f_h(x^i))^2 < \frac{K}{r}$, then go to Step 6.
  - Otherwise unfavorable STOP because convenient solution does not exist for problem (4).
- Step 6.
  - Search a common direction of minimization s solving the following system of inequalities $s(\nabla f_{i_j}(x^i))^T < 0, j = 1,2,...,p$
  - If there is no such a direction s then let s be the direction of minimization for the criterion which realized the largest nonconcordance in Step 5 (that is for the criterion h^* for which $\max_{h=1,2,..,p}\left(z_h - z_h\left(x^i\right)\right)^2 = \left(z_{h^*} - z_{h^*}\left(x^i\right)\right)^2$).
  - Compute

$$\lambda_{\max} = \min_{i|a^i s > 0} \left\{ \frac{b_i - a^i x^i}{a^i s} \right\}.$$

  and according to ERA go to Step 7 with $x^{i+1} = x^i + \lambda_{\max} s$.
- Step 7. If $S\left(x^{i+1}\right) < S\left(x^i\right)$ then go to Step 4 with $i = i+1$. Otherwise, compute $x^{i+1} = x^i + \lambda s$ for $\lambda < \lambda_{\max}$ and return to the test of Step 7.

5. Computational results

In order to illustrate our method let us consider the following multiple objective linear fractional programming problem.

$$\text{(5)} \qquad \text{" min "} \left(z(x) = \left(\frac{x_1}{x_2}, -\frac{x_1}{x_3}, -\frac{x_2 + x_3}{x_2 + 1} \right) \right)$$

subject to

$$\text{(6)} \qquad 1 \leq x_i \leq 4, i = 1,2,3.$$

The marginal solutions of problem (5)-(6) are described by the table.

	x	$z_1(x)$	$z_2(x)$	$z_3(x)$
A	$(4,1,1)$	4	$\boxed{-4}$	-1
B	$(4,4,1)$	1	$\boxed{-4}$	-1
C	$(1,4,1)$	$\boxed{0.25}$	-1	-1
D	$(1,4,4)$	$\boxed{0.25}$	-0.25	-1.6
E	$(1,1,4)$	1	-0.25	$\boxed{2.5}$
F	$(4,1,4)$	4	-1	$\boxed{2.5}$

The values resulted after the execution of the algorithm are grouped into the following table.

i	x^i	$z_1(x^i)$	$z_2(x^i)$	$z_3(x^i)$	$S(x^i)$
0	$(1,1,4)$	1	-0.25	-2.5	14.625
1	$(1,4,4)$	0.25	-0.25	-1.6	14.872
2	$(1,3,4)$	0.33	-0.25	-1.75	14.632
3	$(1,2,4)$	0.5	-0.25	-2	14.375
4	$(1.667,4,2.667)$	0.417	-0.625	-1.33	12.780
5	$(1.571,3.333,2.5)$	0.471	0.629	-1.346	12.470
6	$(3.96,3.333,1)$	1.188	-3.96	-1	3.132
7	$(4,4,1)$	1	-4	-1	2.812
8	$(1,2.5,2.5)$	0.4	-0.4	-1.429	14.27
9	$(3,3.5,1.5)$	0.857	-2	-1.111	6.28
10	$(3.8,3.9,1.1)$	0.97	-3.45	-1.02	2.99
11	$(3.9,3.95,1.05)$	0.98	-3.71	-1.01	2.83
12	$(3.98,3.99,1.01)$	0.99	-3.94	-1.002	3.23
13	$(2,3,3)$	0.66	-0.66	-1.5	12.32
14	$(3.8,3.9,1.2)$	0.97	-3.16	-1.04	3.32
15	$(3.98,3.99,1.02)$	0.99	-3.90	-1.004	3.23
16	$(3.6,3.9,1.1)$	0.92	-3.27	-1.02	3.13
17	$(3,3.9,1.1)$	0.76	-2.7	-1.02	3.05
18	$(3.5,4,1)$	0.875	-3.5	-1	2.89
19	$(3.8,4,1)$	0.95	-3.8	-1	2.78

The best evaluation of the sum $S(x) = \sum_{h=1}^{3} (z_h - z_h(x))^2$ is x^{19}.

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A parallel procedure for the exhaustive numerical solution of the polynomial equations

Ioan Dziţac, Grigor Moldovan, Horea Oros

Abstract. For the exhaustive numerical determination of the real roots of an algebraic equation with real coefficients the Sturm's sequence method can be used. The complex roots of such an equation can be derived by means of the Bairstrow method. Both methods are of a serial type. In this paper a parallel strategy for simultaneous computation of the real and complex roots of algebraic equations with real coefficients, based on a combination of iterative and randomized methods, is proposed.

1. Introduction

Consider the normalized algebraic polynomial equation, of degree n (with $p[0] = 1$), with real coefficients and without zero-roots (with $p[n] \neq 0$)

$$pol[n](x) = \sum_{i=0}^{n} p[i] * x^{n-i} = 0$$

For the determination of the complex roots we take into account the fact that in the complex plane their afixes are situated between the circles of radius $r = rinf$ and $R = rsup$ (fig. 1). Thus, if ζ is a complex root of $pol[n]$ and $a = re\zeta$ and $b = im\zeta$ are the real and the imaginary part of ζ respectively, then we have $a, b \in (-R, R)$ with $R^2 > a^2 + b^2 > r^2$. Since the coefficients of the equation are real numbers, the nonreal roots are complex conjugated. Therefor it is sufficient to determine the roots with the afixes situated in the upper plane, delimited by the x-axis.

Using the polar coordinates

$\rho = ro =$ polar radius

$\theta = theta =$ polar argument

we have

$a = \rho \cos\theta,\ b = \rho \sin\theta,$ with $\theta \in [0, \pi]$ and $\rho \in (r, R)$.

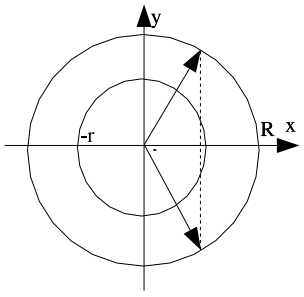

Figure 1. The relationship between coordinates.

In the search for the complex roots we use p processors in parallel: $P[1], P[2], ..., P[p]$. The interval $[0, \pi]$ is splitted in p subintervals of length π/p. Thus, the processor $P[k]$ searches the solution for

$\theta[k] \in [(k-1)\pi/p,\ k\pi/p], k = 1...p$

If one complex root is determined and we denote it by (a, b), then we can reduce the degree of the equation by 2, by the dividing $pol[n](x)$ by $x^2 - 2re[k]x + re[k] * re[k] + im[k] * im[k]$. The remainder of the division has the form: $r0[k]x + r1[k]$.

For this division we use the processor DIV. At the input we have the dividend and the divisor, as vectors of coefficients, while at the output we have the quotient and the remainder (which has to be equal to zero).

2. Processor for the random generation of the polar radius

The processor GDP will randomly generate the values of $\rho = ro$, using the formula: $\rho = r + (R - r) * rnd$, where rnd is a pseudorandom number generator.

Figure 2. GDP processor.

3. Processor for the random generation of the polar argument

The processor GAP randomly generates the values $\theta[k]$ using the formula $\theta[k] = (k-1)\pi/p + (\pi/p) * rnd = theta[k]$.

Figure 3. GAP processor

4. Processor for the determination of the real and imaginary parts

The proccesor PRI uses the transformations:

$$re = ro\cos(theta),\ im = ro\sin(theta),$$

$$with\ theta \in [0, \pi] and\ ro \in (rinf, rsup).$$

FIGURE 4. PRI processor

5. Parallel procedure for the derivation of the non-real roots

```
procedure com_par(n,pol,eps,p);
select eps,p;

begin
    a:=rinf(n,pol);b:=rsup(n,pol)
    poz:=vars(a,b); neg:=vars(-b,-a);
    com:=n-poz-neg;
    m:=0;

    while (m <> com/2)
        for k=1 to p do in parallel (asynchronous)
            ro[k]:=rinf+(rsup-rinf)*rnd;
            teta[k]:=(k-1)*pi/p+rnd*pi/p;
            re[k]:=ro[k]*cos(teta[k]);
            im[k]:=ro[k]*sin(teta[k]);
            call div(pol,imp[k],n,2);
            if(r0[k]=0 or r0[k]<=eps) and
                 (r1[k]=0 or r1[k]<=eps)
            then m:=m+1;
    endif;
        endfor;
    endwhile;
end.
```

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A parallel algorithm for the solution of the polynomial equations possesing real roots only

Ioan Dziţac, Simona Dziţac, Mădălina Văleanu

Abstract. The problem of parallel localizing and computation of the real roots of polynomial equations which possess only real roots is solved. In order to separate the real roots of a polynomial some serial methods exists, e.g. the Sturm's sequence method [1], Vincent's method [2]. In this paper we present a parallel algorithm based on the Newton's method and some parallel concepts from [3] and [4]

1. Introduction

There are practical situations in which we know that an algebraic polynomial equation has only real roots (e.g., in the determination of the eigenvalues of a symmetric matrix, the charateristic equation of a real matrix has this property).

Consider the normalized algebraic polynomial equation, of degree n, with real coefficients, (with $p[0] = 1$) and having all real roots not equal to zero, (with $p[n] \neq 0$):

$$pol[n](x) = \sum_{i=0}^{n} p[i] * x^{n-i} = 0 \tag{1}$$

2. Parallel model of the Newton's simplified model

If the number of positive roots of the equation (1) is *poz* then the number of negative roots is $neg = n - poz$. Let us assume that we have at our disposal a computation system consisting of $p + q$ processors. We distribute p processors for the searching of the positive roots and q processors for the searching of the negative roots. The numbers p and q are proportional to the numbers *poz* and *neg*. We call this method the *news_par*.

In order to search the negative roots, the *news* processes will be activated with an initial value equal to the one at the right end of the interval. It will be stoped when the number of the founded positive roots *nrdap* will be equal to *poz*.

For searching the negative roots, the *news* processes will be activated with an initial value equal to the left margin of the interval and will be stoped when the number of found negative roots *nradn* will be equal to *neg*. These initializations are performed in order to ensure the convergence.

The main program Prog. 1 activates in parallel the $p + q$ processors. The processors function asynchronously in parallel.

Prog. 1 news_par.

```
procedure news_par(n, pol[n], eps)
select eps;
begin
    a := rinf(n, pol[n], eps);
    b := rsup(n, pol[n], eps);
    call vars(vs);
    poz := n - poz;
    neg := n - poz;
cobegin

for k = 1 to p do in parallel asynchronous
    a[k] := a + (k-1)*(b-a)/p;
    b[k] := a + k*(b-a)/p;
    radp := news(n, pol[n], a[k], b[k], b[k], eps);
    if (nradp = poz) then exit;
```

```
endfor;

for k = 1 to q do in parallel asynchronous
    a[k] := a + (k-1)*(b-a)/q;
    b[k] := a + k*(b-a)/q;
    radn := news(n, pol[n], -b[k], -a[k], -b[k], eps);
    if (nradn = neg) then exit;
endfor;
coend;
end.
```

3. Randomized filtered asynchronous method

An algebraic polynomial equation is equivalent to a system of nonlinear algebraic equations, by Newton's relation of recurence

$$x_1 + x_2 + ... + x_n = b_1,\ (x_1)^2 + (x_2)^2 + ... + (x_n)^2 = b_2,\ \ldots\ (x_1)^n + (x_2)^n + ... + (x_n)^n = b_n. \tag{2}$$

The values of $b_1, b_2 \ldots, b_n$ can be determined by recurence, in terms of the coefficients of the initial equation. We solve the system 2 using MIARF algorithm described in [3] pp. 100-114.

By using the Newton's recurence relation

$$S_i = (x_i)^i + (x_2)^i + \cdots + (x_n)^i = b_i, \tag{3}$$

we may form the partial series (asyncronous parallel)

$$x_1^{k+1} = f_i(x^k) = x_1^k + b_1 - S_1^k \tag{4}$$

$$x_{2i+1}^{k+1} = f_i(x^k) = \sqrt[2i+1]{b_{2i+1} + x_{2i+1}^k - S_{2i+1}^k}\ \textit{for the components with odd index} \tag{5}$$

$$x_{2i}^{k+1} = f_{2i}(x^k) = s\sqrt[2i]{b_{2i} + x_{2i}^k - S_{2i}^k},\ \textit{for the components with even index} \tag{6}$$

where

$$s = \begin{cases} 1,\ if\ x_{2i}\ is\ positive \\ -1,\ if\ x_{2i}\ is\ negative \end{cases} \tag{7}$$

If the number of positive roots of equation (1) is poz, then the number of negative roots is $neg = n - poz$.

Now we yield a procedure for the determination of the number of positive and negative roots, by means of the sign variation occuring in the Sturm's method.

Procedures for computing the number of positive and negative roots.

```
procedure nradr(n, pol[n], eps);
select eps;
begin
    a := rinf(n, pol[n], eps)
    b := rsup(n, pol[n], eps)
    call vars(vs);
    poz := vars(a) - vars(b);
    neg := n - poz;
end.
```

Assume that we have at our disposal a computing system consisting of at least $n = poz + neg$ processors (otherwise we have to redistribute proportionally the computing tasks).

For symmetry we may consider:

- positive roots: $xp[i],\ i = 1 \dots poz$
- negative roots: $xn[j],\ j = neg \dots n$

We use two procedures of parallel MIARF type, which communicate and interchange data while executing, by means of the host processors' memory, asynchronously. The master processor P manages the parallel asynchronous execution of the processes $MIARF\_POZ$ and $MIARF\_NEG$; these processes are parallel.

The processor P receives the subvectors consisting of the positive components and negative components respectively and it combines those subvectors to obtain the vector of the current approxiamtion, at each step of the asynchronous iteration.

Procedure for computing the positive roots.

```
procedure MIARF_POZ
receive xv from processor P;
begin
for i = 1 to poz do in parallel
    if (ex(xv[i]) > 0)
        then xp[i] := rad[i](ex(xv[i]))
        else xp[i] := rand(rinf, rsup);
    endif
    send xp[i] to processor P;
endfor;
wait message from processor P;
end.
```

Procedure for computing the negative roots.

```
procedure MIARF_NEG
receive xv from processor P;
begin
for j = poz to n do in parallel
    if (j is even) then
        if (expression under square root, ex(xv[i]), is positive)
            then xn[i] := -rad[i](ex(xv[j]))
            else xn[j] := rand(-rsup, rinf);
        endif;
    else xn[j] := rad[j](ex(xv[j]));
    endif;
    send xp[j] to processor P;
endfor;
wait mesaj from processor P;
end.
```

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The construction of completed skew group rings

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Abstract. The existence under natural restrictions of a completed skew group ring $R * G$ of a given profinite ring R with identity and a profinite group G is proved.

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1. The completed group ring

By a *profinite ring* (*group*) we mean a compact totally disconnected ring (group). By $\mathbb{N}$ denote the set of all natural numbers and by $\mathbb{N}^*$ the set $\mathbb{N}\setminus\{0\}$. By $\operatorname{Aut} R$ it is denoted the group of all automorphisms of a ring R.

An important construction in the ring theory and the algebraic number theory is the completed group ring, which associates to each profinite group G and each profinite ring R a profinite ring $R[[G]]$ containing the usual group ring $R[G]$ as a dense subring (see, e.g. [2]).

Let R be a profinite ring with identity and let G be a profinite group. We remind how a totally bounded (i.e., Hausdorff and precompact) ring topology on the group ring $R[G]$ is introduced [1].

Let V be a two sided ideal of R and let N be an invariant subgroup of G. Consider the two sided ideal of the group ring $R[G]$,

$$(V,N) = V[G] + (1-N),$$

where $(1-N)$ is the ideal of $R[G]$ generated by the set $1-N$ and

$$V[G] = \left\{ \sum_{i=1}^{n} v_i g_i : n \in \mathbb{N}^*,\ v_1,...,v_n \in V,\ g_1,...,g_n \in G \right\}.$$

Since $\cap(V,N) = 0$, where V runs over all open ideals of R and N runs over all open invariant subgroups of G, the family $\{(V,N)\}$, where V runs over all open ideals of R and N runs over all open invariant subgroups of G, gives a totally bounded topology $\mathfrak{T}$ on $R[G]$.

The completion of the topological ring $(R[G],\mathfrak{T})$ is called the *completed group ring*. The completed group ring is denoted by $R[[G]]$.

Remark 1. *This construction of completed group ring is equivalent to the construction given in* [2], *through the inverse limit of compact rings* $R[G/N]$, *where* N *runs over all open invariant subgroups of* G.

Remark 2. *If* $(R,\mathfrak{T}_1)$ *is a profinite ring,* $(G,\mathfrak{T}_2)$ *is a profinite group and* $\mathfrak{T}$ *is the above constructed ring topology on* $R[G]$, *then* $\mathfrak{T}|R = \mathfrak{T}_1$ *and* $\mathfrak{T}|G = \mathfrak{T}_2$.

2. The completed skew group ring

Let R be a ring with identity, let G be a group and let $\sigma : G \to \operatorname{Aut} R$ be a group homomorphism. If $g \in G$ and $r \in R$ denote $r^g = \sigma(g)(r)$. The *skew group ring* $R * G$ is defined as the free left R-module with G as a free generating set. The multiplication on $R * G$ is defined distributively by using the following rule

$$(r_1 g_1) \cdot (r_2 g_2) = r_1 r_2^{g_1} g_1 g_2 ,$$

for all $r_1, r_2 \in R$ and $g_1, g_2 \in G$.

Remark 3. *If $\sigma(g) = \mathrm{id}_R$ for all $g \in G$, then $R * G$ coincides with the group ring $R[G]$.*

Remark 4. *For each $r, r_1, r_2 \in R$ and $g, g_1, g_2 \in G$ we have:*

a) $0^g = 0;\ 1^g = 1;\ r^1 = r;$
b) $(r_1 + r_2)^g = r_1^g + r_2^g;$
c) $(r_1 \cdot r_2)^g = r_1^g \cdot r_2^g;$
d) $r^{g_1 g_2} = (r^{g_2})^{g_1}$.

Let R be a profinite ring with identity and let G be a profinite group. If V is an open ideal of R and if N is an open subgroup of G, consider the subgroup of $R * G$,

$$(V, N) = V * G + (1 - N)_l ,$$

where $(1 - N)_l$ is the left ideal of $R * G$ generated by the set $1 - N$ and

$$V * G = \left\{ \sum_{i=1}^{k} v_i g_i : k \in \mathbb{N}^*;\ v_1, ..., v_k \in V;\ g_1, ..., g_k \in G \right\} .$$

Obviously, $V * G$ is a right ideal of $R * G$.

Remark 5. *If $V^g \subseteq V$, for all $g \in G$, then $V * G$ is a two-sided ideal of $R * G$.*

Indeed, if $r \in R$, $v \in V$ and $g, g' \in G$, then

$$(rg)(vg') = rv^g gg' \in V * G.$$

Remark 6. *If $N \subseteq \ker \sigma$, then $(1 - N)_l$ is a two-sided ideal of $R * G$.*

Indeed, if $r \in R$, $n \in N$ and $g \in G$, then

$$\begin{aligned} (1 - n) \cdot (rg) &= ((1 - n) \cdot r) \cdot g = (r - r^n n) \cdot g = (r - r\, n) \cdot g \\ &= r \cdot (1 - n) \cdot g = rg \cdot \left(1 - g^{-1} n g\right) . \end{aligned}$$

Let $f : R \to R'$ be a ring homomorphism, let $\phi : G \to G'$ be a group homomorphism and let $\sigma : G \to \operatorname{Aut} R$, $\sigma' : G' \to \operatorname{Aut} R'$ be two group homomorphisms. It is obvious that the mapping

$$\Phi : \sum_{i=1}^{k} r_i g_i \longmapsto \sum_{i=1}^{k} f(r_i)\, \phi(g_i)$$

is a group homomorphism of $(R * G, +)$ in $(R' * G', +)$ extending f and ϕ. Denote by V the kernel of f and denote by N the kernel of ϕ.

Lemma 7. $\ker \Phi = (V, N)$.

Proof. Since $\Phi(vg) = f(v)\phi(g) = 0$, for all $v \in V$ and $g \in G$, it follows that $\Phi(\alpha) = 0$, for all $\alpha \in V * G$. If $x = \sum_{i=1}^{k} r_i g_i \in R * G$ and $n \in N$, then

$$\begin{aligned}\Phi(x \cdot (1-n)) &= \Phi\left(\sum_{i=1}^{k} r_i g_i - \sum_{i=1}^{k} r_i (g_i n)\right) \\ &= \sum_{i=1}^{k} f(r_i)\phi(g_i) - \sum_{i=1}^{k} f(r_i)\phi(g_i n) \\ &= 0.\end{aligned}$$

Therefore, $\Phi(\beta) = 0$, for all $\beta \in (1-N)_l$ and thus $\Phi(x) = 0$ for all $x \in (V, N)$. This implies the inclusion $(V, N) \subseteq \ker \Phi$.

Conversely, let $x = \sum_{i=1}^{k} r_i g_i \in \ker \Phi$. We can assume, without lost of generality, that $g_1 N = ... = g_k N$. Then there exists $g \in G$ such that $\phi(g_1) = ... = \phi(g_k) = \phi(g)$. We can choose $n_1, ..., n_k \in N$ such that $g_1 n_1 = ... = g_k n_k = g$. Since $\Phi(x) = 0$, it follows that $f\left(\sum_{i=1}^{k} r_i\right)\phi(g) = 0$. Therefore, $f\left(\sum_{i=1}^{k} r_i\right) = 0$ and thus $\sum_{i=1}^{k} r_i \in V$. It follows that

$$\begin{aligned}x &= \sum_{i=1}^{k} r_i g_i - \sum_{i=1}^{k} r_i g + \sum_{i=1}^{k} r_i g \\ &= \sum_{i=1}^{k} r_i g_i - \sum_{i=1}^{k} r_i g_i n_i + \sum_{i=1}^{k} r_i g \\ &= \sum_{i=1}^{k} r_i g_i \cdot (1-n_i) + \left(\sum_{i=1}^{k} r_i\right) g \\ &\in (V, N).\end{aligned}$$

□

Lemma 8. *If V runs over all open ideals of R and N runs over all open invariant subgroups of G, then $\cap (V, N) = 0$.*

Proof. Let $x = \sum_{i=1}^{k} r_i g_i$ be a nonzero element of $R * G$ (here $r_i \neq 0$, $i = 1, ..., k$). Choose an open ideal V of R such that $r_1, ..., r_k \notin V$ and choose an open invariant subgroup N of G such that $g_i N \neq g_j N$ for all $i, j = 1, ..., k$, $i \neq j$. If $\Phi_{V,N} : R * G \to (R/V) * (G/N)$ is the (group) homomorphism which extends the natural homomorphisms $R \to R/V$ and $G \to G/N$, then

$$\Phi_{V,N}(x) = \sum_{i=1}^{k} (r_i + V) g_i N \neq 0,$$

i.e., $x \notin (V, N)$. □

If A is a cofinite subring of a ring S, then there exists a two-sided cofinite ideal I of S such that $I \subseteq A \subseteq S$ (Theorem of Lewin, [3]). It follows immediately that for every ring S, for every cofinite subring A and for every two-sided ideal I of S, $I \subseteq A$, then exists a cofinite two-sided ideal J of S such that $I \subseteq J \subseteq A \subseteq S$. Moreover, there exists a largest cofinite two-sided ideal J of S such that $I \subseteq J \subseteq A \subseteq S$.

Let now $V^g \subseteq V$, for all $g \in G$. We introduce a totally bounded ring topology on $R * G$ as follows: for each left ideal (V, N) of $R*G$ denote by $\widetilde{(V,N)}$ the largest cofinite two-sided ideal of $R*G$ for which

$$V * G \subseteq \widetilde{(V,N)} \subseteq (V,N) \subseteq R * G.$$

Consider the finite intersections of ideals of type $\widetilde{(V,N)}$ as a fundamental system of neighborhoods of zero for a ring topology $\mathfrak{T}$ on $R * G$. Its completion will be called the *completed skew group ring*.

Theorem 9. *If $V^g \subseteq V$, for all $g \in G$ and for all open ideal V of R, then $(R, \mathfrak{T}_1)$ is a topological subring of $(R * G, \mathfrak{T})$.*

Proof. The inclusion $i : (R, \mathfrak{T}_1) \to (R * G, \mathfrak{T})$ is a topological embedding. Indeed, if $V_1, ..., V_k$ are open ideals of $(R, \mathfrak{T}_1)$ and $N_1, ..., N_k$ are open invariant subgroups of $(G, \mathfrak{T}_2)$, then

$$i(V_1 \cap ... \cap V_k) \subseteq \widetilde{(V_1, N_1)} \cap ... \cap \widetilde{(V_k, N_k)}.$$

□

Theorem 10. *If $V^g \subseteq V$, for all $g \in G$, and for all open ideal V of R and $\operatorname{Im}\sigma$ is finite, then $(R, \mathfrak{T}_1)$ is a topological subring of $(R * G, \mathfrak{T})$ and $(G, \mathfrak{T}_2)$ is a topological subgroup of the group of units of the ring $(R * G, \mathfrak{T})$.*

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Normal forms for nondegenerated Hopf singularities and bifurcations

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Abstract. This aplication - oriented paper reveals some less obvious facts in dynamical systems and bifurcation theory. Systems of two first-order ordinary differential equations (ode) depending on one real parameter are considered. The derivation of the real and complex normal forms of the corresponding vector fields of the Hopf singularity and around the Hopf bifurcation is sketched. The relationship between the corresponding real and complex Liapunov coefficients is shown. The classical Hopf bifurcation results for the linear versus nonlinear case are presented.

1. Normal form at the Hopf singular point

Proposition 1. [1] *Let* $\mathbf{X}(\mathbf{x})$ *be a smooth vector field with a singularity at the origin satisfying* $\det D\mathbf{X}(\mathbf{0}) > 0$, $TrD\mathbf{X}(\mathbf{0}) = 0$. *Then the real normal form of* $\mathbf{X}$ *is given by*

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \sum_{k=2}^{\left[\frac{1}{2}(N-1)\right]} (y_1^2 + y_2^2)^k \left\{ a_k \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + b_k \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \right\} + O\left(|\mathbf{y}|^{N+1}\right), \tag{1}$$

where $\omega = (\det D\mathbf{X}(\mathbf{0}))^{1/2}, N \geq 3$, *the square bracket stands for the integer part and* $a_k, b_k \in \mathbb{R}$.

Proof. Let $D\mathbf{X}(\mathbf{0}) = \mathbf{A}$. Then the eigenvalues of A are $\lambda_{1,2} = \pm i\sqrt{\det \mathbf{A}} = i\omega$ and the corresponding eigenvectors read $\mathbf{u} \pm i\mathbf{v}$. It follows that the real Jordan form of $\mathbf{A}$ is $\mathbf{J} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$, where $\mathbf{P} = (\mathbf{v}|\mathbf{u})$. Denote $\mathbf{x} = (x_1, x_2)^T$, $x_1, x_2 \in \mathbb{R}$, $\mathbf{m} = (m_1, m_2)$, $m_1, m_2 \in \mathbb{N}$, $\mathbf{e}_1 = (1,0)^T$ and $\mathbf{e}_2 = (0,1)^T$. Since $\mathbf{J}$ is not diagonal, the monomials $\mathbf{x}^{\mathbf{m}}\mathbf{e}_i$ are not eigenvectors of $L_{\mathbf{J}}$ and the normal form method is not directly applicable. In order to obtain the diagonal form of $\mathbf{A}$ we use the real linear transformation $\mathbf{x} = \mathbf{P}\mathbf{y}$. Then the linearized equation becomes $\dot{\mathbf{x}} = \mathbf{P}\dot{\mathbf{y}} \Leftrightarrow \mathbf{A}\mathbf{x} = \mathbf{P}\dot{\mathbf{y}} \Leftrightarrow \mathbf{A}\mathbf{P}\mathbf{y} = \mathbf{P}\dot{\mathbf{y}} \Leftrightarrow \dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{y} \Leftrightarrow \dot{\mathbf{y}} = \mathbf{J}\mathbf{y}$. Further, in order to put $\mathbf{J}$ into diagonal form we use the complex linear transformation $\mathbf{y} = \mathbf{P}_{\mathbb{C}}\mathbf{z}$, where $y = (y_1, y_2)^T$, $y_1, y_2 \in \mathbb{R}$, $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 + iy_2 \\ y_1 - iy_2 \end{pmatrix} = \begin{pmatrix} z \\ \overline{z} \end{pmatrix}$ the columns of $\mathbf{P}_{\mathbb{C}}$ are the normalized eigenvectors of $\mathbf{J}$. Then, we obtain $\dot{\mathbf{y}} = \mathbf{P}_{\mathbb{C}}\dot{\mathbf{z}}$. In this way, the linearized equation in y becomes $\mathbf{J}\mathbf{y} = \mathbf{P}_{\mathbb{C}}\dot{\mathbf{z}} \Leftrightarrow \mathbf{J}\mathbf{P}_{\mathbb{C}}\mathbf{z} = \mathbf{P}_{\mathbb{C}}\dot{\mathbf{z}} \Leftrightarrow \dot{\mathbf{z}} = \mathbf{P}_{\mathbb{C}}^{-1}\mathbf{J}\mathbf{P}_{\mathbb{C}}\mathbf{z} \Leftrightarrow \dot{\mathbf{z}} = \mathbf{J}_{\mathbb{C}}\mathbf{z}$. We have $\mathbf{P}_{\mathbb{C}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, $\mathbf{P}_{\mathbb{C}}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, implying $\mathbf{J}_{\mathbb{C}} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}$.

In this way of the nonlinear equation $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ reads by the normal form theorem, it follows that the complex normal form of the nonlinear equation $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ reads

$$\dot{\mathbf{z}} = \mathbf{J}_{\mathbb{C}} \begin{pmatrix} z \\ \overline{z} \end{pmatrix} + \sum_{r=2}^{N} \mathbf{w}_r(\mathbf{z}) + O\left(|\mathbf{z}|^{N+1}\right), \tag{2}$$

where $w_r(z)$ are the resonant terms. Since J_c is diagonal the monomials $\mathbf{z}^{\mathbf{m}}\mathbf{e}_i$, $i = 1,2$ are the eigenvectors of $L_{\mathbf{J}_{\mathbb{C}}}$ and the usual resonance conditions apply yielding the form of $w_r(z)$. This conditions are $m_1\lambda_1 + m_2\lambda_2 = \lambda_i \Leftrightarrow m_1\lambda_1 + (r - m_i)\lambda_2 = \lambda_i \Leftrightarrow m_1(\lambda_1 - \lambda_2) + r\lambda_2 = \lambda_i \Leftrightarrow 2m_1 i\omega - ri\omega = \pm i\omega \Leftrightarrow 2m_1 - r = \pm 1 \Rightarrow r = 2m_1 \mp 1$, therefore the only resonant terms are those which have r an odd number, i.e. $r = 2k + 1$.

a) For $i = 1$ we have $m_1 \cdot i\omega - (2k + 1 - m_1)i\omega = i\omega \Rightarrow m_1 - (2k + 1) + m_1 = 1 \Rightarrow 2m_1 = 2k + 2 \Rightarrow m_1 = k + 1, m_2 = k$.

b) For $i = 2$ we have $m_1 \cdot i\omega - (2k + 1 - m_1)i\omega = -i\omega \Rightarrow m_1 - (2k + 1) + m_1 = -1 \Rightarrow 2m_1 = 2k \Rightarrow m_1 = k, m_2 = k + 1$.

If follows that the resonance condition is satisfied for $\mathbf{m} = (k + 1, k)$, if $i = 1$ ąnd $\mathbf{m} = (k, k + 1)$ if

$i = 2$. Hence, up to some factors, the resonant monomials are given by $z^{k+1}\overline{z}^k\mathbf{e}_1$ and $z^k\overline{z}^{k+1}\mathbf{e}_2$, where $k \in \mathbb{Z}^+$. Correspondingly, (2) becomes

$$\begin{pmatrix}\dot{z}\\ \dot{\bar{z}}\end{pmatrix} = J_{\mathbb{C}}\begin{pmatrix}z\\ \bar{z}\end{pmatrix} + \sum_{k=1}^{\left[\frac{1}{2}(N-1)\right]}\begin{pmatrix}(a_k+ib_k)|\mathbf{z}|^k z\mathbf{e}_1\\ (a_k-ib_k)|\mathbf{z}|^k\bar{z}\mathbf{e}_2\end{pmatrix} + O\left(|\mathbf{z}|^{N+1}\right) \tag{3}$$

where $a_k, b_k \in \mathbb{R}$ are constants.

Let us show that the real form of (3) is (1). To this aim taking into account that $z = y_1 + iy_2$, we have

$$z^{k+1}\overline{z}^k\mathbf{e}_1 = |z|^k z\mathbf{e}_1 = (y_1^2+y_2^2)^{2k}(y_1+iy_2)\begin{pmatrix}1\\0\end{pmatrix} = (y_1^2+y_2^2)^{2k}\begin{pmatrix}y_1+iy_2\\0\end{pmatrix},$$

$$z^k\overline{z}^{k+1}\mathbf{e}_2 = |z|^k\overline{z}\mathbf{e}_2 = (y_1^2+y_2^2)^{2k}(y_1-iy_2)\begin{pmatrix}0\\1\end{pmatrix} = (y_1^2+y_2^2)^{2k}\begin{pmatrix}0\\y_1-iy_2\end{pmatrix}.$$

Since $\mathbf{y} = \mathbf{P}_{\mathbb{C}}\mathbf{z}$, in order to obtain the equation in $\mathbf{y}$ from (3), we multiply (3) by $\mathbf{P}_{\mathbb{C}}$. Then $\mathbf{P}_{\mathbb{C}}(z^{k+1}\overline{z}^k\mathbf{e}_1) = (y_1^2+y_2^2)^{2k}\left[\begin{pmatrix}y_1\\y_2\end{pmatrix} - i\begin{pmatrix}-y_2\\y_1\end{pmatrix}\right]$. Similary $\mathbf{P}_{\mathbb{C}}(z^k\overline{z}^{k+1}\mathbf{e}_2) = (y_1^2+y_2^2)^{2k}\left[\begin{pmatrix}y_1\\y_2\end{pmatrix} + i\begin{pmatrix}-y_2\\y_1\end{pmatrix}\right]$.

Thus, for each $k \in \mathbb{Z}^+$, the set

$$\left\{(y_1^2+y_2^2)^{2k}\begin{pmatrix}y_1\\y_2\end{pmatrix}, (y_1^2+y_2^2)^{2k}\begin{pmatrix}-y_2\\y_1\end{pmatrix}\right\}$$

spans a subspace, $\mathcal{G}^{2k+1}$, of $\mathcal{H}^{2k+1}$, complementary to $L_{\mathbf{J}}(\mathcal{H}^{2k+1})$. Here $\mathcal{H}^{2k+1}$ stands for the linear spare of two dimensional vectors the entries of which are homogeneous polynomials of degree $2k+1$ and $L_{\mathbf{J}}$ is the Poisson-Lie operator.

Moveover, $\mathbf{P}_{\mathbb{C}}$ (3) becomes (1). □

Remark 2. *For $N = 3$, (1) becomes*

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix} = \begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} + (y_1^2+y_2^2)\left\{a_1\begin{pmatrix}y_1\\y_2\end{pmatrix} + b_1\begin{pmatrix}-y_2\\y_1\end{pmatrix}\right\} + O\left(|\mathbf{y}|^4\right). \tag{4}$$

Conclusion 3. *Comparison of the real and complex normal form (1) and (3) respectively shows that the coefficients a_k and b_k in (1) are the real and imaginary part respectively of (3).*

The detailed proof of Proposition 1 provides the method of constructing the normal form (1) of the vector field $\boldsymbol{X}(\boldsymbol{x})$ defining a given vector ode $\dot{\boldsymbol{x}} = \boldsymbol{X}(\boldsymbol{x})$. In other words, by this method we find the coefficients a_k and b_k.

2. Normal form around the equilibrium point

Consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\alpha), \qquad \mathbf{x} = (\mathbf{x}_1,\mathbf{x}_2)^T \in \mathbb{R}^2, \alpha \in \mathbb{R}, \tag{5}$$

with a smooth function f, which at $\alpha = 0$ has the equilibrium $\mathbf{x} = \mathbf{0}$ with the eigenvalues $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$. By the Implicit Function Theorem, the system (5) has a unique equilibrium $\mathbf{x}_0(\alpha)$ in some neighborhood of the origin for all sufficiently small $|\alpha|$, since $\lambda = 0$ is not an eigenvalue of the Jacobian matrix. Therefore, without loss of generality, we may assume that $\mathbf{x} = \mathbf{0}$ is the equilibrium point of the system for $|\alpha|$ sufficiently small. Thus, the system can be written as

$$\dot{\mathbf{x}} = \mathbf{A}(\alpha)\mathbf{x} + \mathbf{F}(\mathbf{x},\alpha), \tag{6}$$

where F is a smooth vector function whose components F_1, F_2 have Taylor expansions in $\mathbf{x}$ starting with at least quadratic terms.

Lemma 4. [3] *By introducing a complex variable $z = x_1 + ix_2$, for sufficiently small $|\alpha|$ system (6) can be written as a single equation*

$$\dot{z} = \lambda(\alpha)z + g(z,\bar{z},\alpha), \tag{7}$$

where $g = O(|z|^2)$ is a smooth function of $(z,\bar{z},\alpha)$.

Remark 5. *The function g can be written as a formal Taylor series in two complex variables (z and $\bar{z}$)*

$$g(z,\bar{z},\alpha)=\sum_{k+l\geq 2}\frac{1}{k!l!}g_{kl}(\alpha)z^k\bar{z}^l. \tag{8}$$

Lemma 6. [3] *For all sufficiently small $|\alpha|$, by an invertible parameter dependent change of variables*

$$z=w+\frac{h_{20}}{2}w^2+h_{11}w\bar{w}+\frac{h_{02}}{2}\bar{w}^2,$$

the equation

$$\dot{z}=\lambda z+\frac{g_{20}}{2}z^2+g_{11}z\bar{z}+\frac{g_{02}}{2}\bar{z}^2+O(|z|^3), \tag{9}$$

where $\lambda=\lambda(\alpha)=\mu(\alpha)+i\omega(\alpha),\mu(0)=0,\omega(0)=\omega_0>0$, and $g_{ij}=g_{ij}(\alpha)$, can be transformed into an equation without quadratic terms

$$\dot{w}=\lambda w+O(|w|^3). \tag{10}$$

Proof. Let $\lambda_1=\lambda$ and $\lambda_2=\bar{\lambda}$ be the eigenvalues of $\mathbf{A}(\alpha)$ and denote $\tilde{h}_{(i,j),k}=\frac{h_{(i,j),k}}{i!j!}, i,j=0,1,2,\ k=1,2$. The eigenvalues of L_A are $\Lambda_{\mathbf{m},i}=m_1\lambda_1+m_2\lambda_2-\lambda_i$, thus we obtain
$\Lambda_{(2,0),1}=2\lambda+0\cdot\bar{\lambda}-\lambda=\lambda\neq 0,$
$\Lambda_{(1,1),1}=1\cdot\lambda+1\cdot\bar{\lambda}-\lambda=\bar{\lambda}\neq 0,$
$\Lambda_{(0,2),1}=0\cdot\lambda+2\bar{\lambda}-\lambda=2\bar{\lambda}-\lambda\neq 0,$
$\Lambda_{(2,0),2}=2\lambda+0\cdot\bar{\lambda}-\bar{\lambda}=2\lambda-\bar{\lambda}\neq 0,$
$\Lambda_{(1,1),2}=1\cdot\lambda+1\cdot\bar{\lambda}-\bar{\lambda}=\lambda\neq 0,$
$\Lambda_{(0,2),2}=0\cdot\lambda+2\bar{\lambda}-\bar{\lambda}=\bar{\lambda}\neq 0.$
Then, by normal form method, Table 1 can be constructed providing the expressions of $h_{\mathbf{m},i}$ in terms of g_{kl}. In fact, we must consider also the transformation $\bar{z}\longleftrightarrow\bar{w}$. Consequently, we obtain the equation in $\dot{\bar{w}}$ corresponding to (10). However, due to the fact that in (3) the equation for $\dot{\bar{z}}$ is the complex conjugate to the equation for $\dot{z}$ and the transformation $\bar{z}\longleftrightarrow\bar{w}$ is the complex conjugate of the transformation $z\longleftrightarrow w$, the equation for $\dot{\bar{w}}$ is the complex conjugate of (10). This is why it is written no longer. In addition, we write $h_{\mathbf{m}}$ for $h_{\mathbf{m},1}$ and $h_{m_1,m_2,2}=\bar{h}_{m_1,m_2,2}$

m_1	m_2	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},1}$	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\tilde{h}_{\mathbf{m},1}$	$\tilde{h}_{\mathbf{m},2}$
2	0	λ	$2\lambda-\bar{\lambda}$	$\frac{g_{20}}{2}$	$\frac{\bar{g}_{20}}{2}$	$\frac{g_{20}}{2\lambda}$	$\frac{\bar{g}_{20}}{2\lambda-\bar{\lambda}}$
1	1	$\bar{\lambda}$	λ	g_{11}	$\bar{g}_{11}$	$\frac{g_{11}}{\bar{\lambda}}$	$\frac{\bar{g}_{11}}{\lambda}$
0	2	$2\bar{\lambda}-\lambda$	$\bar{\lambda}$	$\frac{g_{02}}{2}$	$\frac{\bar{g}_{02}}{2}$	$\frac{g_{02}}{2(2\bar{\lambda}-\lambda)}$	$\frac{\bar{g}_{02}}{2\bar{\lambda}}$

Table 1.

The inverse change of variable is given by the expression

$$w=z-\frac{h_{20}}{2}z^2-h_{11}z\bar{z}-\frac{h_{02}}{2}\bar{z}^2+O(|z|^3),$$

or taking into account Table 1, we have

$$w=z-\frac{g_{20}}{2\lambda}z^2-\frac{g_{11}}{\bar{\lambda}}z\bar{z}-\frac{g_{02}}{2(2\bar{\lambda}-\lambda)}\bar{z}^2+O(|z|^3),$$

while $z=w+\frac{g_{20}}{2\lambda}w^2+\frac{g_{11}}{\bar{\lambda}}w\bar{w}+\frac{g_{02}}{2(2\bar{\lambda}-\lambda)}\bar{w}^2$.

Therefore, by differentiating the last equality we obtain

$$\begin{aligned}
\dot{w} &= \dot{z} - \frac{g_{20}}{2\lambda}2z\dot{z} - \frac{g_{11}}{\bar{\lambda}}(\dot{z}\bar{z} + z\dot{\bar{z}}) - \frac{g_{02}}{2(2\bar{\lambda}-\lambda)}2\bar{z}\dot{\bar{z}} + O(|z|^3) \\
&= \dot{z}(1 - \frac{g_{20}}{\lambda}z - \frac{g_{11}}{\bar{\lambda}}\bar{z}) + \dot{\bar{z}}(-\frac{g_{11}}{\bar{\lambda}}z - \frac{g_{02}}{2(2\bar{\lambda}-\lambda)}\bar{z}) + O(|z|^3) \\
&= [1 - \frac{g_{20}}{\lambda}z - \frac{g_{11}}{\bar{\lambda}}\bar{z}](\lambda z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + O(|z|^3) \\
&\quad +[-\frac{g_{11}}{\bar{\lambda}}z - \frac{g_{02}}{2\bar{\lambda}-\lambda}\bar{z}](\bar{\lambda}\bar{z} + \frac{\bar{g_{20}}}{2}\bar{z}^2 + \bar{g_{11}}z\bar{z} + \frac{\bar{g_{02}}}{2}z^2 + O(|\bar{z}|^3) \\
&= \lambda z - \frac{g_{20}}{2}z^2 - g_{11}\frac{\lambda}{\bar{\lambda}}z\bar{z} - \frac{\lambda g_{02}}{2(2\bar{\lambda}-\lambda)}\bar{z}^2 + O(|z|^3) \\
&= \lambda\left(w + \frac{g_{20}}{2\lambda}w^2 + \frac{g_{11}}{\bar{\lambda}}w\bar{w} + \frac{g_{02}}{2(2\bar{\lambda}-\lambda)}\bar{w}^2\right) - \frac{g_{20}}{2}w^2 \\
&\quad -\frac{\lambda}{\bar{\lambda}}g_{11}(w + \frac{g_{20}}{2\lambda}w^2 + ...)(\bar{w} + \frac{\bar{g_{20}}}{2\bar{\lambda}}\bar{w}^2 + ...) - \frac{\lambda g_{02}}{2(2\bar{\lambda}-\lambda)}\bar{w}^2 + O(|w|)^3 \\
&= \lambda w + O(|w|)^3.
\end{aligned}$$

□

Assuming that we have removed all quadratic terms, let us try to eliminate the cubic terms as well. There exists only one resonant term, as the following lemma shows.

Lemma 7. [3] *For all sufficiently small $|\alpha|$, by an invertible parameter dependent change of variables*

$$z = w + \frac{h_{30}}{6}w^3 + \frac{h_{21}}{2}w^2\bar{w} + \frac{h_{12}}{2}w\bar{w}^2 + \frac{h_{03}}{6}\bar{w}^3,$$

the equation

$$\dot{z} = \lambda z + \frac{g_{30}}{6}z^3 + \frac{g_{21}}{2}z^2\bar{z} + \frac{g_{12}}{2}z\bar{z}^2 + \frac{g_{03}}{6}\bar{z}^3 + O(|z|^4), \tag{11}$$

where $\lambda = \lambda(\alpha) = \mu(\alpha) + i\omega(\alpha), \mu(0) = 0, \omega(0) = \omega_0 > 0$, and $g_{ij} = g_{ij}(\alpha)$, can be transformed into an equation with only one cubic term

$$\dot{w} = \lambda w + c_1 w^2\bar{w} + O(|w|^4), \tag{12}$$

where $c_1 = c_1(\alpha)$.

Proof. The eigenvalues of L_A are $\Lambda_{\mathbf{m},i} = m_1\lambda_1 + m_2\lambda_2 - \lambda_i$, therefore we obtain
$\Lambda_{(3,0),1} = 3\lambda + 0\cdot\bar{\lambda} - \lambda = 2\lambda \neq 0,$
$\Lambda_{(2,1),1} = 2\lambda + 1\cdot\bar{\lambda} - \lambda = \lambda + \bar{\lambda} = 0,$
$\Lambda_{(1,2),1} = 1\cdot\lambda + 2\bar{\lambda} - \lambda = 2\bar{\lambda} \neq 0,$
$\Lambda_{(0,3),1} = 0\cdot\lambda + 3\bar{\lambda} - \lambda = 3\bar{\lambda} - \lambda \neq 0,$
$\Lambda_{(3,0),2} = 3\lambda + 0\cdot\bar{\lambda} - \bar{\lambda} = 3\lambda - \bar{\lambda} \neq 0,$
$\Lambda_{(2,1),2} = 2\lambda + 1\cdot\bar{\lambda} - \bar{\lambda} = 2\lambda \neq 0,$
$\Lambda_{(1,2),2} = 1\cdot\lambda + 2\bar{\lambda} - \bar{\lambda} = \lambda + \bar{\lambda} = 0,$
$\Lambda_{(0,2),2} = 0\cdot\lambda + 2\bar{\lambda} - \bar{\lambda} = \bar{\lambda} \neq 0.$
It follows that the resonant terms are $w\bar{w}^2\mathbf{e}_1$ and $w^2\bar{w}\mathbf{e}_2$ (Table2) and

m_1	m_2	$\Lambda_{\mathbf{m},1}$	$\Lambda_{\mathbf{m},1}$	$X_{\mathbf{m},1}$	$X_{\mathbf{m},2}$	$\tilde{h}_{\mathbf{m},1}$	$\tilde{h}_{\mathbf{m},2}$
3	0	2λ	$3\lambda-\bar{\lambda}$	$\frac{g_{30}}{6}$	$\frac{\bar{g}_{30}}{6}$	$\frac{g_{30}}{12\lambda}$	$\frac{\bar{g}_{30}}{6(3\lambda-\bar{\lambda})}$
2	1	$\lambda+\bar{\lambda}$	2λ	$\frac{g_{21}}{2}$	$\frac{\bar{g}_{21}}{2}$	–	$\frac{\bar{g}_{21}}{4\lambda}$
1	2	$2\bar{\lambda}$	$\lambda+\bar{\lambda}$	$\frac{g_{12}}{2}$	$\frac{\bar{g}_{12}}{2}$	$\frac{g_{12}}{4\bar{\lambda}}$	–
0	3	$3\bar{\lambda}-\lambda$	$2\bar{\lambda}$	$\frac{g_{03}}{6}$	$\frac{\bar{g}_{03}}{6}$	$\frac{g_{03}}{6(3\bar{\lambda}-\lambda)}$	$\frac{\bar{g}_{03}}{12\bar{\lambda}}$

Table 2.

the change of variables $z \longleftrightarrow w$ becomes

$$z = w + \frac{g_{30}}{12\lambda}w^3 + \frac{h_{21}}{2}w^2\bar{w} + \frac{g_{12}}{4\bar{\lambda}}w\bar{w}^2 + \frac{g_{03}}{6(3\bar{\lambda}-\lambda)}\bar{w}^3. \tag{13}$$

The inverse change of variable is given by the expression

$$w = z - \frac{g_{30}}{12\lambda}z^3 - \frac{h_{21}}{2}z^2\bar{z} - \frac{g_{12}}{4\bar{\lambda}}z\bar{z}^2 - \frac{g_{03}}{6(3\bar{\lambda}-\lambda)}\bar{z}^3 + O(|z|^4). \tag{14}$$

Like in the case of Lemma 2, due to the fact the component of (3) along $\mathbf{e}_2$ is the complex conjugate of the component of (3) along $\mathbf{e}_1$, we wrote no longer the complex conjugate of (13) and (14).

Therefore, by differentiating (14), we have

$$\begin{aligned}
\dot{w} &= \dot{z} - \frac{g_{30}}{12\lambda}3z^2\dot{z} - \frac{h_{21}}{2}(2z\dot{z}\bar{z} + z^2\dot{\bar{z}}) - \frac{g_{12}}{4\bar{\lambda}}(\dot{z}\bar{z}^2 + 2z\bar{z}\dot{\bar{z}}) - \\
&\quad - \frac{g_{03}}{6(3\bar{\lambda}-\lambda)}3\bar{z}^2\dot{\bar{z}} + O(|z|^4) \\
&= \lambda z + \left(\frac{g_{30}}{6} - \frac{g_{30}}{4}\right)z^3 + \left(\frac{g_{21}}{2} - \lambda h_{21} - \frac{\bar{\lambda}}{2}h_{21}\right)z^2\bar{z} - \frac{\lambda g_{12}}{4\bar{\lambda}}z\bar{z}^2 + \\
&\quad + \left(\frac{g_{03}}{6} - \frac{\bar{\lambda}g_{03}}{2(3\bar{\lambda}-\lambda)}\bar{z}^3\right) + O(|z|^4) \\
&= \lambda z - \frac{g_{03}}{12}z^3 + \left(\frac{g_{21}}{2} - \frac{2\lambda+\bar{\lambda}}{2}h_{21}\right)z^2\bar{z} - \frac{\lambda}{4\bar{\lambda}}g_{12}z\bar{z}^2 - \frac{\lambda}{6(3\bar{\lambda}-\lambda)}g_{03}\bar{z}^3 + \\
&\quad + O(|z|^4). \qquad (15)
\end{aligned}$$

Substituting (13) in (15) we obtain

$$\dot{w} = \lambda w + \left[\frac{g_{21}}{2} - \frac{2\lambda+\bar{\lambda}}{2}h_{21}\right]w^2\bar{w} + O(|z|^4). \tag{16}$$

In order to obtain a transformation that is smoothly dependent on α, set $h_{21} = 0$ which results in $c_1 = \frac{g_{21}}{2}$. In fact, in the normal form theory, in (13) the term containing h_{21} is considered any longer. (Indeed, formally, by the homological equation we would have $h_{21} = \frac{X_{21,1}}{\Lambda_{21,1}} = \frac{X_{21,1}}{0}$) □

Remark 8. *The remaining cubic $w^2\bar{w}$-term is a resonant term. As expected, its coefficient is the same as the coefficient of the cubic term $z^2\bar{z}$ in the original equation.*

The other resonant therm occurs in the equation complex conjugate to (12).

We now combine the two previous lemmas to obtain, in a similar way,

Lemma 9 (Poincaré normal form for the Hopf bifurcation)**.** [3] *For all sufficiently small $|\alpha|$, be an invertible parameter dependent change of variables, smoothly depending on the parameter,*

$$z = w + \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2 + + \frac{h_{30}}{6}w^3 + \frac{h_{21}}{2}w^2\bar{w} + \frac{h_{12}}{2}w\bar{w}^2 + \frac{h_{03}}{6}\bar{w}^3,$$

the equation

$$\dot{z} = \lambda z + \sum_{2\leq k+l\leq 3} \frac{1}{k!l!} z^k \bar{z}^l + O(|z|^4), \tag{17}$$

where $\lambda = \lambda(\alpha) = \mu(\alpha) + i\omega(\alpha), \mu(0) = 0, \omega(0) = \omega_0 > 0$, and $g_{ij} = g_{ij}(\alpha)$, can be transformed into an equation with only one cubic term

$$\dot{w} = \lambda w + c_1 w^2 \bar{w} + O(|w|^4), \tag{18}$$

where

$$c_1 = c_1(\alpha) = \frac{g_{20}g_{11}(2\lambda + \bar{\lambda})}{2|\lambda|^2} + \frac{|g_{11}|^2}{\lambda} + \frac{|g_{02}|^2}{2(2\lambda - \bar{\lambda})} + \frac{g_{21}}{2}. \tag{19}$$

Let us now come back to the real state functions. We have $w = y_1 + iy_2$, hence $\dot{w} = \dot{y}_1 + i\dot{y}_2$, such that 18) reads

$$\begin{aligned} \dot{y}_1 + i\dot{y}_2 &= (\mu + i\omega)(y_1 + iy_2) + (Re\ c_1 + iIm\ c_1)(y_1^2 + 2iy_1y_2 - y_2^2)(y_1 - iy_2) \\ &= (\mu y_1 - \omega y_2 + y_1^3 Re\ c_1 + y_1y_2^2 Re\ c_1 - y_1^2y_2 Im\ c_1 - y_2^3 Im\ c_1) \\ &\quad + i(\omega y_1 + \mu y_2 + y_1^2y_2 Re\ c_1 + y_2^3 Re\ c_1 + y_1^3 Im\ c_1 + y_1y_2^2 Im\ c_1), \end{aligned}$$

or equivalently,

$$\begin{aligned} \dot{y_1} &= \mu y_1 - \omega y_2 + (y_1^2 + y_2^2)[y_1 Re\ c_1 - y_2 Im\ c_1], \\ \dot{y_2} &= \omega y_1 + \mu y_2 + (y_1^2 + y_2^2)[y_2 Re\ c_1 + y_1 Im\ c_1], \end{aligned}$$

or, finally,

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + (y_1^2 + y_2^2) \left\{ a_1(\alpha) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + b_1(\alpha) \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \right\} + O\left(|\mathbf{y}|^4\right). \tag{20}$$

Conclusion 10. *The coefficients a_1 and b_1 in the complex normal form (18) are related to the coefficients of the real normal form (20) by the relations $a_1 = Re\ c_1$ and $b_1 = Im\ c_1$. However, in spite of their label of normal forms, the equations (and the corresponding vector fields) (18) and (20) are not the simplest in their equivalence class. Indeed, the question is about the dependence on α through the coefficients, and not about the dependence on z because all the nonresonant terms were removed. So far no assumption about the sign of $c_1(\alpha)$ was made. Moreover, $c_1(\alpha)$ can vanish. For the Hopf bifurcation we are interested in the case of constant sign for $c_1(\alpha)$. In addition, we want to obtain the true normal form depending on a single new parameter and this parameter occurs as a coefficient. In [3] a method to obtain this true normal form is presented. In fact, this true normal form is the versal unfolding of (2) or (3), i.e. of the normal form of the singularity.*

Lemma 11. [3] *Let equation (18) be written as*

$$\frac{dw}{dt} = (\mu(\alpha) + i\omega(\alpha))w + c_1(\alpha)w|w|^2 + O(|w|^4), \tag{21}$$

where (H_1) $\mu(0) = 0$ and (H_2) $\omega(0) = \omega_0 > 0$. If (H_3) $\frac{d\mu}{d\alpha}(0) \equiv \mu'(0) \neq 0$ and (H_4) $Rec_1(0) \neq 0$, then (20) can be transformed by a parameter dependent linear function transformation, a time rescaling, and a nonlinear time reparametrization into an equation of the form

$$\frac{du}{d\theta} = (\beta + i)u + su|u|^2 + O(|u|^4), \tag{22}$$

where u is a new complex state function and θ and β are the new time and parameter respectively, and $s = sign\ Re\ c_1(0) = \pm 1$.

Therefore the dynamical systems corresponding to (21) and (22) are topologically aquivalent. Letting $u = u_1 + iu_2$, the real form of the (22) reads

$$\frac{d}{d\tau}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \pm (u_1^2 + u_2^2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + O(\|\mathbf{u}\|^4). \tag{23}$$

The coefficient $c_1(\alpha)$ of the resonant term in (21) is called the *first complex Liapunov coefficient*. In proving Lemma 5, some intermediate real normal form of (21) occurs. Its coefficients are expressed in terms of the new parameter

$\beta \equiv \beta(\alpha) = \dfrac{\mu(\alpha)}{\omega(\alpha)}$ and of $d_1(\beta) = \dfrac{c_1(\alpha(\beta))}{\omega(\alpha(\beta))}$, $e_1(\beta) = Im\ d_1(\beta)$, $l_1(\beta) = Re\ d_1(\beta) - \beta e_1(\beta)$. Since $l_1(\beta)$ is the coefficient of the resonant term in this form, it is called the *first real Liapunov coefficient*.

Theorem 12. [3] *Assume that for all sufficiently small $|\alpha|$ the two-dimensional system (5) with smooth f has the equilibrium $\mathbf{x} = \mathbf{0}$ with the corresponding eigenvalues $\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha)$, where (H_1) and (H_2) hold. If the nondegeneracy conditions (H_3) and (H_4) are satisfied then there are invertible changes of functions and parameter and a time reparametrization transforming (5) into (23).*

Condition (H_4) reads, equivalently, as $l_1(0) \neq 0$.

Theorem 13. [3] ***(Topological normal form for the Hopf bifurcation)*** *Any generic two-dimensional system (5) having at $|\alpha| = 0$ the equilibrium $\mathbf{x} = \mathbf{0}$ with the eigenvalues $\lambda_{1,2}(0) = \pm i\omega(0), \omega(0) > 0$, is locally topologically equivalent near the origin to one the following true normal forms*

$$\frac{d}{d\theta}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \pm (u_1^2 + u_2^2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{24}$$

3. The classical Hopf bifurcation

One among the two simplest real normal forms for the two- dimensional Hopf bifurcations is associated with the system

$$\begin{cases} \dot{x} = \beta x - y + x(x^2 + y^2), \\ \dot{y} = x + \beta y + y(x^2 + y^2), \end{cases} \tag{25}$$

where we used the notation: $u_1 = x$ si $u_2 = y$. This bifurcation (i.e. this dynamical system) has for $\beta = 0$, the equilibrium point $\mathbf{x} = \mathbf{0}$.

The linearized system around this point reads $\dot{x} = \beta x - y$, $\dot{y} = x + \beta y$ and it has the eigenvalues $\lambda_{1,2} = \beta \pm i$. The corresponding static bifurcation diagram for $(25)_1$ is given in fig.1.

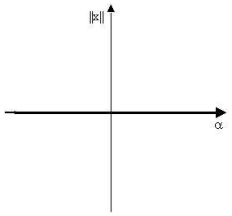

FIGURE 1

The dynamic bifurcation diagrams in the linear case and nonlinear case are represented in figs.2a and 2b respectively.

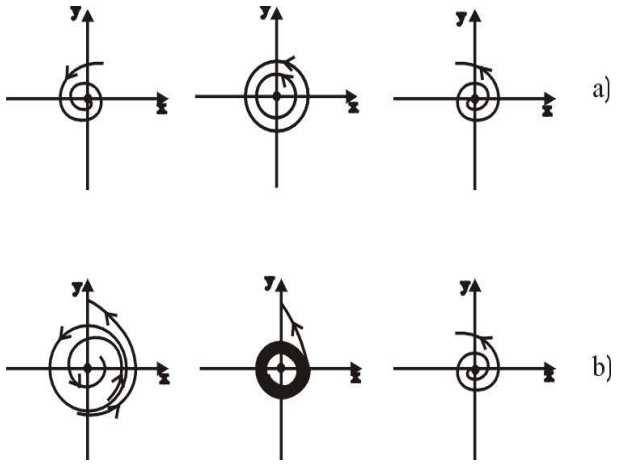

FIGURE 2 $\beta < 0$ $\beta = 0$ $\beta > 0$

They show that as the decreasing parameter β crosses the critical value $\beta = 0$, a repulsive limit cycle occurs subcriticaly (i.e. for $\beta < 0$). Simultaneously the equilibrium situated at the origin of the (x,y) phase space becomes attractive. The limit cycle corresponds to periodic dynamics. The appearance of periodic motions is referred to as the Hopf bifurcation phenomenon. The point $(\beta, x, y) = (0, 0, 0)$ is called the *Hopf bifurcation point.* The same name is beared by the dynamical system associated with $(25)_1$ for $\beta = 0$. The value $\beta = 0$ referred to as the Hopf bifurcation value. The singularity $(x, y) = (0, 0)$ of the vector field defining $(25)_1$ for $\beta = 0$ is called the *Hopf singularity.* The same name is given to the vector field itself.

More generally, all dynamical system generated by (20) and having the eigenvalues $\mu(\alpha) + i\omega(\alpha)$, corresponding to equilibrium point situated at the origin, satisfying the properties in Lemma (5) are referred to Hopf bifurcations. Indeed, by Lemma (5), they are topologically equivalent to (22) and, by Theorem 2, they are topologically equivalent to (20). More precisely we have

Definition 14. *The equilibrium* $\mathbf{x} = \mathbf{0}$ *is a* ***Hopf bifurcation point*** *for (5) if in a neighborhood of* $\alpha = 0$ *the normal form of the vector field in (5) reads*

$$F(\mathbf{y}, \alpha) = \begin{pmatrix} \mu(\alpha) & -\omega(\alpha) \\ \omega(\alpha) & \mu(\alpha) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + (y_1^2 + y_2^2) \left\{ a_1 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + b_1 \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \right\}.$$

Definition 15. *The equilibrium* $\mathbf{x} = \mathbf{0}$ *is a* ***Hopf bifurcation point*** *for (5) if conditions* $(H_1) - (H_4)$ *hold. The singularity is the vector field possessing a Hopf singular point for a given value of the parameter (here* $\alpha = 0$*) while the bifurcation refers to the dynamical scheme i.e. the unfolding of the singularity for values of* α *around* $\alpha = 0$.

Theorem 16 (Hopf bifurcation)**.** [2] *Suppose that conditions* (H_3) *and* (H_4) *hold. Then there are* $\sigma > 0$ *and a neighborhood* U *of* $(x, y) = (0, 0)$ *such that:*
(i) if $|\alpha| < \sigma$ *and* $Re(c_1)\mu'(0)\alpha < 0$ *the system*

$$\dot{x} = f(x, y, \alpha), \ \dot{y} = g(x, y, \alpha), \tag{26}$$

where $x, y, \alpha \in \mathbb{R}$*, has exactly one limit cycle inside* U*;*
(ii) if $|\alpha| < \sigma$ *and* $Re(c_1)\mu'(0)\alpha \geq 0$ *the system (26) has no periodic orbits inside* U.
Moreover, the limit cycle is attractive (repulsive) if $Re(c_1) < 0$ $(Re(c_1) > 0)$*, and it tends to the equilibrium* $(0, 0)$ *as* $\alpha \to 0$.

Recall that $(Re(c_1(0)) = a_1(0)$. Hence so far all our reasonings concern the case $a_1(0) \neq 0$. In this case we say that the Hopf bifurcation is nondegenerated; otherwise Hopf bifurcation is degenerated. It is interesting to mention that Theorem 3 asserts the existence of a precisely one limit cycle. In the degenerated case, a results analogous to Theorem 3 asserts [2] the existence of exactly k limit cycles if the real form of the Liapunov coefficients are $a_1(0) = a_2(0) = ... = a_{k-1}(0) = 0$ and $a_k(0) \neq 0$.

However, in general, this happens if several parameters exist. If $a_k(0) = 0$, for every $k \geqslant 1$, then [2], as α crosses the value $\alpha = 0$, the Hopf bifurcation degenerates to a **nonlinear centre**, i.e. the phase space of this dynamical system consists of closed trayectories.

Conclusion 17. *The eigenvalues of the linearized system and the real parts of the Liapunov coefficients of the normal form at the Hopf singularity determine the type of the corresponding (degenerated or nondegenerated) Hopf bifurcation and, in particular, the number of the limit cycles.*

Conclusion 18. [4] *To a linear centre, a nonlinear centre, or one limit cycle or several limit cycles can correspond.*

Remark 19. *Theorems asserting the existence of one or several limit cycles are important because, in general, by other methods it is very difficult to prove the existence of periodic solutions of o.d.e.s.*

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Singularities of the secular equation in a Bénard magnetic problem

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Abstract. This paper belongs to a series [1]-[10] in which we studied the magnetic Bénard problem. In applyng the Budiansky-DiPrima method in order to determine the neutral curve we looked for the eigenvalues of the governing eigenvalue problem in the range of regular values for the associated secular equation. Herein for the case of the absence of Hall currents we prove that the singular values are eigenvalues and we determine the corresponding neutral curve. If the Hall current is present there to singular values no eigenvalue correspond.

1. Introduction

The linear Bénard magnetic problem received considerable attention in the literature [1]-[9] (for a list of references we recommend [9]). We treated this problem in several physical situations: without Hall and ion-slip effects; with Hall efects; with Hall and ion-slip effects; for rigid walls; for free surfaces; with additional effects (e.g. porosity, fluid mixture). Our methods were analytical or numerical. The analytical methods were direct or based on series. In this last case the Budiansky-DiPrima method was employed. In all these papers we were interested in bounds of stability or instability in the form of criteria or neutral curves.

In this paper we analyze a problem avoided so for, namely we investigate if the excepted values (singular for the secular equation) are eigenvalues of the governing problem. We mention that the eigenvalues are the zeros of the secular equation and they define curves separating the stability and instability regions in the parameter space. Among these curves, that one corresponding to lowest values for a main parameter (e.g. Rayleigh number) is taken as neutral.

After a brief presentation of the perturbation problem (Section 2) and of the neutral curves for regular values (Section 3), in Section 4 we treat the case when Hall currents are absent and in Section 5 the case when they are present. Only the case of the even velocity and temperature vector fields and odd magnetic field as functions of the vertical coordinate is considered.

2. Governing problem

Consider a horizontal layer of a viscous fluid bounded by free surfaces $z = \pm 0.5$, subject to a vertical constant magnetic field, characterized by the Hartmann numbers M, a vertical upwards temperature gradient defined by the Rayleigh number R and Hall currents (characterized by β_H). Then in the nondimensional form, the eigenvalue problem governing the linear stability of a mechanical equilibrium of this layer to normal mode perturbations of wave number a reads

$$\begin{aligned}
&(D^2 - a^2)K + DW - \beta_H DX = 0,\\
&(D^2 - a^2)Z + M^2 DX = 0,\\
&(D^2 - a^2)^2 W + M^2 D(D^2 - a^2)K - R\tfrac{P_m}{P_r} a^2 \Theta = 0,\\
&(D^2 - a^2)X + DZ + \beta_H D(D^2 - a^2)K = 0,\\
&\tfrac{P_m}{P_r}(D^2 - a^2)\Theta + W = 0,
\end{aligned} \tag{1}$$

$$W = D^2 W = K = DX = DZ = \Theta = 0 \text{ at } z = \pm 0.5, \tag{2}$$

Assume that W, Θ, and X are even functions of z while K and Z are odd. Here W, Z, K, X are the components of the velocity, rotor of velocity, magnetic field, rotor of magnetic field in the vertical direction respectively and Θ is the temperature. Moreover, P_r and P_m are Prandtl and magnetic Prandtl numbers.

The two-point problem (1), (2) contains six positive parameters. For physical reasons we choose R as the eigenvalue. Therefore the eigenvalue depends on five parameters. The stability and instability regions are separated, in the parameter space, by the curve $R = R_1(a, M, \beta_H, P_m, P_r)$ where R_1, is the smallest eigenvalue.

In our investigations we solved problem (1), (2) by expansion in Fourier series on the total sets $\{E_{2n-1}\}$ and $\{F_{2n-1}\}$, where $E_{2n-1} = \sqrt{2}\cos((2n-1)\pi z)$, $F_{2n-1} = \sqrt{2}\sin((2n-1)\pi z)$. Since the expansion functions do not satisfy all boundary conditions (2) some constraints occur.

3. Natural curves for the regular case

In Fourier coefficients problem (1), (2) reads

$$
\begin{array}{l}
-A_nK^o_{2n-1} - (2n-1)\pi W^e_{2n-1} + \beta_H(2n-1)\pi X^e_{2n-1} = \\
= 2\sqrt{2}(-1)^n(\alpha_6 - \beta_H\alpha_4), \\
-A_nZ^o_{2n-1} - M^2(2n-1)\pi X^e_{2n-1} = 2\sqrt{2}(-1)^n\alpha_4 M^2, \\
A_n^2W^e_{2n-1} - M^2(2n-1)\pi A_nK^o_{2n-1} - \frac{P_m}{P_r}Ra^2\Theta^e_{2n-1} = \\
= 2\sqrt{2}(-1)^n(2n-1)\pi\alpha_6 M^2, \\
-A_nX^e_{2n-1} + (2n-1)\pi Z^o_{2n-1} - \beta_H(2n-1)\pi A_nK^o_{2n-1} = \\
= 2\sqrt{2}(-1)^n(2n-1)\pi(\alpha_6\beta_H + \alpha_4), \\
-A_n\frac{P_m}{P_r}\Theta^e_{2n-1} + W^e_{2n-1} = 0,
\end{array} \tag{3}
$$

where $\alpha_6 = DK^o(0.5)$, $\alpha_4 = X^e(0.5)$, while the constraints become

$$
\sum_{n=1}^{\infty}\left[2\sqrt{2}\alpha_4 - (-1)^{n+1}(2n-1)\pi X^e_{2n-1}\right] = 0,\ \sum_{n=1}^{\infty}(-1)^{n+1}K^o_{2n-1} = 0. \tag{4}
$$

Let Δ_n be the associate Cramer determinant for (15) and denote $X^e_{2n-1} = \Delta_{4n}/\Delta_n$ and $K^o_{2n-1} = \Delta_{5n}/\Delta_n$. Suppose that $\Delta_n \neq 0$. Then we have

$$
\begin{array}{l}
\Delta_n = \frac{P_m}{P_r}A_n\{(Ra^2 - A_nH_n)(H_n + \beta_H^2L_n) + M^2\beta_H^2L_n^2\}, \\
\Delta_{4n} = 2\sqrt{2}(-1)^{n+1}\frac{P_m}{P_r}(2n-1)\pi\alpha_4[(Ra^2 - A_nH_n) + M^2\beta_H^2A_n^2L_n], \\
\Delta_{5n} = 2\sqrt{2}(-1)^n\frac{P_m}{P_r}\alpha_4a^2\beta_HA_n(Ra^2 - A_n^3) - 2\sqrt{2}(-1)^n\frac{\Delta_n}{A_n}\alpha_6,
\end{array} \tag{5}
$$

leading to the secular equation

$$
\sum_{n-1}^{\infty}\frac{A_n^2(Ra^2 - A_nH_n)}{A_n\{(Ra^2 - A_nH_n)(H_n + \beta_H^2L_n) + M^2\beta_H^2L_n^2\}} = 0, \tag{6}
$$

where we used the notation $L_n = A_n(A_n - a^2)$, $H_n = A_n^2M^2(A_n - a^2)$.

Truncating the series (6) to the first term and assuming that $\beta_H \neq 0$, $\Delta_n \neq 0$ we obtained the eigenvalue $R = A_1H_1a^{-2}$. Introducing the notation $X_n = Ra^2 - A_nH_n$, this eigenvalue corresponds to $X_1 = 0$.

Still in the hypotheses $\beta_H \neq 0$, $\Delta_n \neq 0$, for the first two terms of the series (6) the secular equetion becomes

$$
X_1^2 + X_1[M^2\beta_H^2B_+ - Q] - M^2\beta_H^2QD = 0, \tag{7}
$$

where $X_1 = Ra^2 - A_1H_1$, $B_+ = \frac{A_2L_1^2 + A_1L_1^2}{A_1G_2 + A_2G_1}$, $Q = H_2A_2 - H_1A_1$, $D = \frac{A_1L_2^2}{A_1G_2 + A_2G_1}$ and $G_n = H_n + \beta_H^2L_n$. Correspondingly, there are two eigenvalues $R_1 = (A_1H_1 + X_1^{(1)})a^{-2}$ and $R_2 = (A_1H_1 + X_1^{(2)})a^{-2}$, where $X_1^{(1),(2)}$ are the roots of (7) and $X_1^{(2)} < 0 < X_1^{(1)}$. The number of eigenvalues R increases with the number of retained terms in (6).

From physical point of view we are interested in the neutral curve $R = R(a, M)$ corresponding to the smallest eigenvalue. In our cases, this curve is $R = R_2(a, M)$. Since $X_1^{(2)} < 0$ it follows that R_2 is smaller than the value corresponding to one term, i.e. the neutral curve is that from the case of two terms in (6).

In order to find if the Hall currents are stabilizing or destabilizing we must compare this curve with that from the case $\beta_H = 0$. In taking $\beta_H = 0$ in (6) we obtain

$$(8) \qquad \sum_{n=1}^{\infty} \frac{A_n^2(Ra^2 - A_nH_n)}{An\{(Ra^2 - A_nH_n)(H_n + \beta_H^2 L_n)\}},$$

or, if $Ra^2 - A_nH_n \neq 0$,

$$(9) \qquad \sum_{n=1}^{\infty} \frac{A_n}{H_n + \beta_H^2 L_n} = 0.$$

This relation does not hold since every term of the series is positive. If $Ra^2 - A_nH_n = 0$ then (8) in singular. This singularity is due to the fact that in this case $\Delta_n = 0$. Because (8) has no meaning, the case $Ra^2 = A_nH_n$ must be treated separately.

Let us remark that in the case $\beta_H = 0$, Δ_n spitted in a product of two factors and corresponded to the decoupling of the system (3).

4. Neutral curves for the singular case $\beta_H = 0$, $\Delta_n = 0$

If $\beta_H = 0$, the system (3) decouples in the systems

$$(10) \qquad \begin{array}{l} -A_nZ^o_{2n-1} - M^2(2n-1)\pi X^e_{2n-1} = 2\sqrt{2}(-1)^n\alpha_4 M^2, \\ -A_nX^e_{2n-1} + (2n-1)\pi Z^o_{2n-1} = 2\sqrt{2}(-1)^n\alpha_4(2n-1)\pi, \end{array}$$

and

$$(11) \qquad \begin{array}{l} -A_nK^o_{2n-1} - (2n-1)\pi W^e_{2n-1} = 2\sqrt{2}(-1)^n\alpha_6, \\ A_n^2W^e_{2n-1} - M^2(2n-1)\pi A_nK^o_{2n-1} - \frac{P_m}{P_r}Ra^2\Theta^o_{2n-1} = \\ = 2\sqrt{2}(-1)^n(2n-1)\pi\alpha_6 M^2, \\ -\frac{P_m}{P_r}A_n\Theta^e_{2n-1} + W^e_{2n-1} = 0. \end{array}$$

Correspondingly $\Delta_n = [H_n]\left[\frac{P_m}{P_r}A_n(Ra^2 - A_nH_n)\right] = \Delta_n' \cdot \Delta_n''$ where Δ_n' and $\Delta_n'' = H_n$ are the Crammer determinants of these systems. From the first system it follows $X^e_{2n-1} = -\frac{(M^2+A_n)(2n-1)\pi}{H_n}2\sqrt{2}(-1)^n\alpha_4$ such that the first constraint (4) becomes $2\sqrt{2}\alpha_4 \sum\limits_{n=1}^{\infty}\left[1 + \frac{(M^2+A_n)(A_n-a^2)}{H_n}\right] = 0$ which is not valid (because each term is positive) unless $\alpha_4 = 0$. But, if $\alpha_4 = 0$ it follows that X^e_{2n-1} and $Z^o_{2n-1} = 0$. Hence in order to see if there is some eigenvalue we must study the system (11).

Thus, assume first $\Delta_n'' \neq 0$, i.e. $Ra^2 - A_nH_n \neq 0$ for every $n \in \mathbb{N}^*$. Then $\Delta_n'' = \frac{P_m}{P_r}A_n(Ra^2 - A_nH_n)$, $W^e_{2n-1} = \Theta^e_{2n-1} = 0, K^o_{2n-1} = \frac{2\sqrt{2}(-1)^{n+1}\alpha_6}{A_n}$ such that the second constraint (4) becomes the relation $2\sqrt{2}\alpha_6 \sum \frac{1}{A_n} = 0$ implying $\alpha_6 = 0$ and, therefore, $W^e_{2n-1} = \Theta^e_{2n-1} = 0$. Hence $R \neq A_nH_na^{-2}$ for every $n \in \mathbb{N}^*$ cannot be an eigenvalue. It remains to study the case $R = A_mH_ma^{-2}$. In this case $\Delta_n'' = 0$ while $\Delta_m'' \neq 0$ if $n \neq m$. In the case $n \neq m$ we have $\Delta_n'' = \frac{P_m}{P_r}A_n(A_mH_m - A_nH_n)$, $W^e_{2n-1} = \Theta^e_{2n-1} = 0$, $K^o_{2n-1} = \frac{2\sqrt{2}(-1)^{n+1}\alpha_6}{A_n}$. In the case $n = m$ the equations in (11) are not linearly independent. Indeed, $M^2(2n-1)\pi(11)_1 - [A_n^2 + M^2(A_n - a^2)](11)_3 = (11)_2$. Therefore for $n = m$ we consider the system $(11)_{1,3}$ the solution of which are $W^e_{2m-1} = \frac{P_m}{P_r}A_m\Theta^e_{2m-1}$, $K^o_{2m-1} = -\frac{P_m}{P_r}(2m-1)\pi\Theta^e_{2m-1} - \frac{2\sqrt{2}(-1)^m\alpha_6}{A_m}$. In this way, the constraint $(4)_2$ becomes $\sum\limits_{n=1}^{\infty}\left[\frac{2\sqrt{2}(-1)^{n+1}\alpha_6}{A_n}\right] + (-1)^m\frac{P_m}{P_r}(2m-1)\pi\Theta^e_{2m-1} = 0$. Taking into account that $\sum\limits_{n=1}^{\infty}\frac{1}{A_n} = \frac{1}{4a}th\frac{a}{2}$ it follows that $\alpha_6 = (-1)^{m+1}cth\frac{a}{2}\frac{P_m}{P_r}a\sqrt{2}(2m-1)\pi\Theta^e_{2m-1}$. Since two by parts integrations yield $\int\limits_{-0.5}^{0.5} shaz\sin(2n-1)\pi z dz = \frac{2a(-1)^{n+1}}{A_n}ch\frac{a}{2}$, it means that $\sum\limits_{n=1}^{\infty}\frac{2a(-1)^{n+1}}{A_n}F_{an-1}$

is the Fourier series expansion on the set $\{F_{2n-1}\}$ of the function $\frac{ashaz}{ch\frac{a}{2}}$. Taking into account the value of α_6 it follows that $K^o(z) = -\sqrt{2}\frac{P_m}{P_r}(2m-1)\pi\Theta^e_{2m-1}\left[(-1)^m\frac{sh(az)}{sh\frac{a}{2}} + \sin((2m-1)\pi z)\right]$.

Moreover we have $W^e(z) = \sqrt{2}\frac{P_m}{P_r}A_m\Theta^e_{2m-1}\cos((2m-1)\pi z)$ and $\Theta^e(z) = \sqrt{2}\Theta^e_{2m-1}\cos((2m-1)\pi z)$. Consequently for any value $R = A_mH_ma^{-2}$ the system (11) has this nontrivial solution and, therefore, this value is an eigenvalue. Of course, it should suffice the fact that $\Theta^e(z)$ is not vanishing; we gave the expressions of $W^e(z)$ and $K^o(z)$ since they are useful if we want to know the corresponding eigensolution. Thus the neutral curve is $R_1 = A_1H_1a^{-2}$ i.e. $R_1 = (\pi^2+a^2)[(\pi^2+a^2)^2 + M^2\pi^2]a^{-2}$ and it corresponds to the perturbations $W^e(z) = \sqrt{2}\frac{P_m}{P_r}\pi\Theta^e_1\cos(\pi z)$, $\Theta^e(z) = \sqrt{2}\Theta^e_1\cos(\pi z)$, $K^e(z) = -\sqrt{2}\frac{P_m}{P_r}\pi\Theta^e_1\left[\sin(\pi z) - \frac{shaz}{sh\frac{a}{2}}\right]$.

5. The singular case $\beta_H \neq 0,\ \Delta_n = 0$

In this case

$$Ra^2 = A_mH_m - \beta_H^2M^2\frac{L_m^2}{H_m+\beta_H^2L_m}. \tag{12}$$

In addition, for $m = n$,

$$\frac{(3)_1M^2(2m-1)\pi H_m + (3)_2(2m-1)\pi\beta_HM^2(A_m-a^2)}{H_m+\beta_H^2L_m} +$$
$$+\frac{\beta_HM^2L_m}{H_m+\beta_H^2L_m}(3)_4 - (3)_3 + \frac{\alpha_4\beta_Ha^2A_m(2m-1)\pi M^2}{H_m+\beta_H^2L_m} = 0$$

if in equations (3) the expressions in the right-hand sides were passed in the left-hand sides. It follows that equation $(3)_3$ is a linear combination of $(3)_{1,2,4,5}$ if $\alpha_4 = 0$. Therefore assume that $\alpha_4 = 0$ Then the system (3) for $n = m$ is consistent and the system $(3)_{1,2,4,5}$ has the solutions $W^e_{2n-1} = \frac{P_m}{P_r}A_m\Theta^e_{2m-1}$, $Z^o_{2n-1} = -\frac{\frac{P_m}{P_r}(2m-1)\pi\Theta^e_{2m-1}\beta_HM^2L_m^2}{A_m(H_m+\beta_H^2L_m)}$, $X^e_{2m-1} = \frac{\frac{P_m}{P_r}\beta_H\Theta^e_{2m-1}L_m^2}{H_m+\beta_H^2L_m}$, $K^e_{2m-1} = \frac{-\frac{P_m}{P_r}(2m-1)\pi\Theta^e_{2m-1}H_m}{H_m+\beta_H^2L_m} - \frac{2\sqrt{2}(-1)^m\alpha_6}{A_m}$.

For $m \neq n$ formulae (5) still hold, of course for Ra^2 given by (12) and $\alpha_4 = 0$. In addition, direct computations give $W^e_{2n-1} = \Theta^e_{2n-1} = X^e_{2n-1} = Z^o_{2n-1} = 0$, while $K^e_{2n-1} = -\frac{2\sqrt{2}(-1)^n\alpha_6}{A_n}$. Then restriction $(4)_1$ implies $\Theta^e_{2m-1} = 0$ and $(4)_2$ implies $\alpha_6 = 0$, whence the solution of (3) is trivial and, consequently, R given by (12) is not an eigenvalue for the problem (1), (2).

6. Conclusions

The governing eigenvalue problem in perturbations was investigated in two cases in which the associated secular equation is singular. By Budiansky-DiPrima method the problem was reduced for $n \in \mathbb{N}^*$ to an algebraic affine system in the Fourier coefficients the Cramer determinant Δ_n of which is singular. The singularities are of the from $R = R_m(a, M, \beta_H)$, where R_m is a function involving $m \in \mathbb{N}^*$, and they are the same as for the secular equation.

The Fourier coefficients were determined separately for $n \neq m$ and for $n = m$ and then we introduced the results in the constraints. In this way we found that, in the absence of the Hall effect, $R = R_m$ was an eigenvalue while in the other case when the Hall effect was present no such eigenvalues exist.

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Fixed points for mappings on probabilistic 2 metric spaces

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Abstract. In this paper some connections of different classes of probabilistic 2-metric spaces together with connections of these spaces with 2-metric spaces are studied. Some fixed point theorems for set-valued mappings on probabilistic 2-metric spaces are proved.

Keywords: probabilistic 2-metric space, fixed point

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Throughout this paper we assume that $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+ = [0, \infty)$ and $I = [0, 1]$.

Definition 1. *A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function, if it is nondecreasing, left-continuous with* $\inf F(t) = 0$, *and* $\sup F(t) = 1$.

In the following we denote by $\mathcal{D}$ the set of all distribution functions. $\mathcal{D}^+ = \{F : F \in \mathcal{D}, F(t) = 0,$ for all $t \leq 0\}$ is the set of all distance distribution functions, that is the set of all distribution functions associated with non-negative, one dimensional random variables. H_a denotes the function

$$H_a(t) = \begin{cases} 1, & \text{if} \quad t > a; \\ 0, & \text{if} \quad t \leq a. \end{cases}$$

Definition 2. *Let S be a nonempty set. The pair (S, d) is called a 2-metric space if the mapping $d : S \times S \times S \to \mathbb{R}$ satisfies the following conditions:*

(M_1) *for any $x, y \in S, x \neq y$ there exists a $z \in S$ such that $d(x, y, z) \neq 0$;*
(M_2) *$d(x, y, z) = 0$, if at least two of three points x, y, z are equal;*
(M_3) *$d(x, y, z) = d(x, z, y) = d(y, z, x)$;*
(M_4) *$d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z), \forall x, y, z, u \in S$.*

In recent decades an important progress has been made in the theory and applications of 2-metric spaces [3-4], [23-24].

Definition 3. *A probabilistic 2-metric space (briefly, a P-2-M space) is a triple $(S, \mathcal{F}, \tau)$, where S is a nonempty set whose elements are the points of the space, $\mathcal{F}$ is a function from $S \times S \times S$ into $\mathcal{D}^+, \mathcal{F}(x, y, z)$ will be denote by $F_{x,y,z}$ and τ is a tetrahedral function such that the following conditions are satisfied for all $x, y, z, u \in S$:*

(P_1) *For each pair of distinct points x, y in S there exists a point z in S such that $F_{x,y,z} \neq H_0$.*
(P_2) *$F_{x,y,z} = H_0$ if at least two of x, y, z are equal;*
(P_3) *$F_{x,y,z} = F_{x,z,y} = F_{y,z,x}$;*
(P_4) *$F_{x,y,z} \geq \tau(F_{x,y,u}, F_{x,u,z}, F_{u,y,z})$.*

If (P_1) , (P_2) and (P_3) are satisfied then $(S, \mathcal{F})$ is called a probabilistic 2-semimetric space (briefly, a P-2-SM space).

In the study of the P-2-M space [5-9] we have deduced that the following functions $\tau : \mathcal{D}^+ \times \mathcal{D}^+ \times \mathcal{D}^+$ defined by $\tau(F, G, H) = \tau_1(F, \tau_1(G, H))$, where τ is a triangle function [2,20] enssure appropriate properties for a probabilistic formulation of the inequality (M_4), called the tetrahedral inequality in a 2-metric space. This is the reason for that we call the function τ the *tetrahedral function* and the inequality (P_4) the *probabilistic tetrahedral inequality.*

Similarly, if T_1 is a t-norm, then the function $T : I \times I \times I \longrightarrow I$ defined by $T(a, b, c) = T_1(a, T_1(b, c))$ is a called a th-norm.

Now, let us consider the following inequality:

$$(P_5) \quad F_{x,y,z}(t) \geq T(F_{x,y,u}(t_1), F_{x,u,z}(t_2), F_{u,y,z}(t_3)) \quad \forall \quad t_1, t_2, t_3 \in \mathbb{R}^+ \quad t_1 + t_2 + t_3 = t.$$

If $(S, \mathcal{F})$ is a P-2-SM space and (P_5) is satisfied, then $(S, \mathcal{F}, T)$ is called a P-2-M-space of Menger's type or simply a 2-Menger space.

Remark 4. *In [1] probabilistic tetrahedral inequality in 2-Menger spaces was defined with t-norms apparently more general. In [16] it was proved that those are, in fact, th-norms.*

Proposition 5. *If T is a left continuous th-norm and τ_T is the tetrahedral function defined by*

$$\tau_T(F, G, H)(t) = \sup_{t_1+t_2+t_3<t} T(F(t_1), G(t_2), H(t_3)) \quad \forall t_1, t_2, t_3, t \in \mathbb{R}^+,$$

then $(S, \mathcal{F}, \tau)$ is a P-2-M space iff $(S, \mathcal{F}, T)$ is a 2-Menger space.

Definition 6. *Let $(S, \mathcal{F})$ be a P-2-SM space such that*

$$(P_6) \quad \left.\begin{array}{l} \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0, \\ F_{x,y,u}(\varepsilon_1) > 1 - \varepsilon_1, \\ F_{x,u,z}(\varepsilon_2) > 1 - \varepsilon_2, \\ F_{u,y,z}(\varepsilon_3) > 1 - \varepsilon_3 \end{array}\right\} \Rightarrow F_{x,y,z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) > 1 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3).$$

Then $(S, \mathcal{F})$ is called a probabilistic 2-metric space of Hikcs' type (briefly, H-P-2-M space) [11-13]. Topological properties of probabilistic 2-metric spaces have been studied in [5-8] and also in [1].

Proposition 7. *Let $(S, \mathcal{F}, \tau)$ be a P-2-M space, where τ is a continuous th-function. Then $(S, \mathcal{F}, \tau)$ becomes a Hausdorff space in the topology $\mathcal{T}(F)$ induced by the family of neighbourhoods*

$$U = \{U(\varepsilon, A) : \varepsilon > 0, A \subset S\},$$

where A is a nonempty and finite subset of S and

$$U(\varepsilon, A) = \{x, y) \in S \times S : F_{x,y,a}(\varepsilon) > 1 - \varepsilon, a \in A\}$$

The connection of a H-P-2-M space and a 2-Menger space is given by:

Proposition 8. *If $(S, \mathcal{F}, T)$ is a 2-Menger space and*

$$(T_1) \quad T(a, b, c) = T_m(a, b, c) = Max\{a + b + c - 2, 0\},$$

then $(S, \mathcal{F})$ is a H-P-2-M space.

Proof. Assume that $F_{x,y,u}(\varepsilon_1) > 1 - \varepsilon_1, F_{x,u,z}(\varepsilon_2) > 1 - \varepsilon_2, F_{u,y,z}(\varepsilon_3) > 1 - \varepsilon_3$ If (T_1) holds, then $F_{x,y,z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = Max\{1 - \varepsilon_1 + 1 - \varepsilon_2 + 1 - \varepsilon_3, 0\} > 1 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ and (P_6) follows. □

Remark 9. *As for th-norms $T(a, b, c) = Min\{a, b, c\}$ and $T(a, b, c) = Prod(a, b, c) = a \cdot b \cdot c$ we have $Min \geq T_m$, $Prod \geq T_m$, the conclusion of the proposition follows.*

Theorem 10. *Let $(S, \mathcal{F})$ be a probabilistic 2-metric space and $d(x, y, z) = \sup\{\varepsilon : \varepsilon \in [0, 1)$ and $F_{x,y,z}(\varepsilon) \leq 1 - \varepsilon\}$ Then :*

(D_1) *$d(x, y, z) < t$ iff $F_{x,y,z}(t) > 1 - t$;*
(D_2) *d is a 2-metric on S;*
(D_3) *d is compatible for the topology $\mathcal{T}(\mathcal{F})$;*
(D_4) *$(S, \mathcal{F})$ is complete iff (S, d) is complete;*
(D_5) *if $f : S \times S \to S$ and $0 \leq k < 1$*

then the following contraction conditions are equivalent:

(CH_1) *$t > 0, F_{x,y,a}(t) > 1 - t$ implies $F_{fx,fy,a}(kt) > 1 - kt$;*
(C_1) *$d(fx, fy, a) \leq kd(x, y, a)$.*

Proof. (D_1) if $1 < t$, then $d(x,y,z) \leq 1 < t$ and $F_{x,y,z}(t) \geq 0 > 1-t$. Suppose that $d(x,y,z) < \delta < t$. Then $F_{x,y,z}(t) \geq F_{x,y,\delta}(\delta) > 1-\delta > 1-t$. Conversely suppose that $F_{x,y,z}(t) > 1-t$, where $0 < t \leq 1$. We shall show that there exists δ_0 such that $0 < \delta_0 < t$ and $d(x,y,z) \leq \delta_0 < t$. Let us suppose the contrary, that is $d(x,y,z) > \delta$ for each $\delta < t$. Then $F_{x,y,z}(\delta) \leq 1-\delta$ for each $\delta < 0$. This implies $\lim_{\delta \nearrow t^-} F_{x,y,z}(\delta) = F_{x,y,z}(t) \leq \lim_{\delta \nearrow t^-}(1-\delta) = 1-t$, that is a contradiction. Hence, there exists $\delta > 0$ such that $d(x,y,z) \leq \delta < t$. Thus (D_1) is proved.

(D_2) Let x, y be two distinct points in S. Since there exists a point z in S such that $F_{x,y,z} \neq H_0$, it follows that there exists $\varepsilon > 0$ such that $F_{x,y,z}(\varepsilon) = \delta$ with $0 \leq \delta < 1$. Let us consider that $\delta_1 = 1-\delta$ and $\varepsilon = \min\{\varepsilon, \delta_1\}$. Then $F_{x,y,z}(\varepsilon_1) \leq \delta = 1-\delta_1 \leq 1-\varepsilon_1$. This implies $d(x,y,z) \geq \varepsilon_1 > 0$. Thus (M_1) follows.

(D_3) Let x, y, z in S be such that $x = y$. Then $F_{x,y,z} = H_0$. This means that $F_{x,y,z}(\varepsilon) > 1-\varepsilon$ for each $\varepsilon > 0$. Hence $d(x,y,z) = \sup\{0\} = 0$. From (P_3), follows (M_3).

Let us now show that d satisfies the tetrahedral inequality in a 2-metric spaces (M_4).

Suppose that $d(x,y,u) < \varepsilon_1, d(x,u,z) < \varepsilon_2, d(u,y,z) < \varepsilon_3$. Then there exists $\delta_i, i = 1,2,3$ such that $d(x,y,u) < \delta_1 < \varepsilon_1, d(x,u,z) < \delta_2 < \varepsilon_2, d(u,y,z) < \delta_3 < \varepsilon_3$. Theorem 1 implies that $F_{x,y,u}(\delta_1) > 1-\delta_1, F_{x,u,z}(\delta_2) > 1-\delta_2, F_{u,y,z}(\delta_3) > 1-\delta_3$. From (P_6) it follows that $F_{x,y,z}(\delta_1+\delta_2+\delta_3) > 1 - d(x,y,z) \leq \delta_1+\delta_2+\delta_3 \leq \varepsilon_1+\varepsilon_2+\varepsilon_3$. Thus the tetrahedral inequality (M_4) follows (D_1) shows that d is compatible with $\mathcal{T}(\mathcal{F})$ whence (M_4).

(D_5) Suppose that $d(fx,fy,a) \leq kd(x,y,a)$ and $F_{x,y,a}(t) > 1-t$. Then $d(x,y,a) < t$ and $d(fx,fy,a) > kt$. By (D_1) we have $F_{fx,fy,a}(kt) > 1-kt$. If (C_1) holds, let $\varepsilon > 0$ be given, $t = d(x,y,a) < \varepsilon$. Then $d(x,y,a) = t - \varepsilon < t$ implies $F_{x,y,a}(t) > 1-t$ and, by (C_1), we have $F_{fx,fy,a}(kt) > 1-kt$. Thus we have $d(fx,fy,a) < kt < k(d(x,y,a)+\varepsilon) = kd(x,y,a) + k\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $d(fx,fy,a) \leq kd(x,y,a)$ holds. □

The property (D_5) shows that the contraction condition (CH_1) can be used to translate fixed point theorems from 2-metric space to P-2-M space and conversely.

Proposition 11. *Let ($S, \mathcal{F}$) be a P-2-SM space. If there is a compatible complete 2-metric d on $\mathcal{T}(\mathcal{F})$ such that $d(x,y,z) < t$ iff $F_{x,y,z}(t) > 1-t$, then $(S,\mathcal{F})$ is a H-P-2-M - space.*

Theorem 12. *Let $(S,\mathcal{F})$ be a H-P-2-M space and let $f : S \longrightarrow S$ Assume that $k : [0,\infty) \longrightarrow [0,\infty)$ $k(0) = 0, k$ is strictly increasing and $\lim_{\varepsilon \to 0^+} k(t+\varepsilon) = k(t)$. Then:*

(C_2) $d(f(x),f(y),a) \leq k[d(x,y,a)]$ iff

(CH_2) $t > 0, F_{x,y,a}(t) > 1-t$ implies $F_{f(x),f(y),a}(k(t)) \geq 1-k(t)$.

The study of fixed points for mapping defined on probabilistic metric spaces (PM-spaces) has been initiated by V.M.Sehgal in [22], where a type of probabilistic contraction was introduced. Another type of probabilistic contraction has been introduced by T.L. Hieks in [10-12].

In [21] it is proved that a contraction in the Sehgal sense is not necessary a contraction in sense Hicks and that a Hicks' type contraction is not necessary a contraction of Sehgal's type.

The two types of contractions have also been studied in the framework of probabilistic 2-metric structures [8-9]. The above theorems show that Hicks'type contractions have a better connection with the 2-metrics compatible with the topology generated by probabilistic 2-metrics.

The probabilistic distance from a point $x \in S$ to a subset $C \subset S$ is the following distribution function $F_{x,C,a}(t) = \sup_{y \in C} F_{x,y,a}(t)$ for each a in S.

Definition 13. *A subset $C \subset S$ is said to be probabilistic proximinal if, for any $x, a \in S$, there exists $y \in C$, such that $F_{x,C,a}(t) = F_{x,y,a}(t)$, for all $t \in \mathbb{R}^+$. Denote by PR(S) the set of all probabilistic proximinal subsets of S. One can easily see that $CP(S) \subset PR(S) \subset CL(S)$ whenever $(S,\mathcal{F},T)$ is a 2-Menger space under a continuous th-norm.*

Let f be a set valued function defind on S into $PR(S)$. By an orbit $O_f(x)$ of $x \in S$, generated by f, we mean the sequence $(x_n)_{n \geq 0}$ for which $x_0 = x$ and $x_n \in f(x_{n-1}), n \geq 1$.

Definition 14. *A set- valued function f from S in $P(X)$ is said to be orbitally upper semicontinuous at the point s if, for each orbit of x, $O_f(x) = (x_n)_{n\geq 0}$ which converges to s, we have*

$$(SC) \quad F_{s,f(s),a}(t) \geqslant \overline{\lim_{n\to\infty}} F_{x_n,f(x_n),a}(t)$$

for each $a \in S$ and $t \in \mathbb{R}^+$.

Theorem 15. *Let $(S, \mathcal{F}, T)$ be a complete 2-Menger space under a continuous th-norm T, $T \geq T_m$ and let f be a set-valued mapping of S into $PR(S)$. If there exists a point $x \in S$ and $k \in [0,1)$ such that the orbit $O_f(x) = (x_n)_{n\geq 0}$ satisfied the condition*

$$(RC) \quad t > 0 \quad \text{and} \quad F_{x_{n-1},x_n,a}(t) > 1 - t \Rightarrow F_{x_n,x_{n+1},a}(kt) > 1 - kt$$

for all $n \in N$ and $a \in S$, then:

a) the sequence $(x_n)_n \geq 0$ converges to a points $s \in S$;

b) the point s is a fixed point of f iff f is arbitrary upper semicontinuous at s.

Remark 16. *Theorem 3 can be extended without difficulty to common fixed points for the family of set valued mappings.*

Remark 17. *One can see that the two statements (a) and (b) of Theorem 3 are relatively independent. The first states that the sequence $O_f(x)$ is convergent and the second states that the limit s of the sequence $O_f(x)$ is a fixed point of f. Thus, the condition from (b) can enssure the existence of a fixed point of a mapping in other contraction conditions.*

Corollary 18. *Let (S, d) be a complete and bounded 2-metric space and let f be a mapping of S into PR(S). If for $k \in (0,1)$ there exists $x \in S$ such that an orbit $O_f(x) = (x_n)_{n\in N}$ satisfies the contraction condition*

$$d(x_n, x_{n+1}, a) < k(d(x_{n-1}, x_n, a),$$

for any $n \in N$ and $a \in S$, then:

a) the sequence $O_f(x)$ converges to a point $s \in S$,

b) s is a fixed point of f iff f is orbitally lower semicontinuous at s.

Proof. If we put $F_{x,y,z}(t) = H_0(t - d(x, y, z))$ then $(S, \mathcal{F}, \min)$ is a complete 2- Menger space. □

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Riemann surface approach to bound and resonant states: Exotic poles and resonant states

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Abstract. An approach to bound and resonant states in scattering by a central potential $gV(r)$, based on the construction of the Riemann surface of the pole function k=k(g), is presented. New classes of poles and resonant states are identified and their properties are studied.

By using an usual potential we are looking for the possibility to generate a class of exotic resonant states (ERS) that could be candidates for parent quasimolecular states, i.e. resonant states which have the wave function localized in the region of the barrier, particularly stable with respect to the dissolution into the neighboring compound nucleus resonant states. In order to do this we have to identify all classes of resonant states and to study their properties.

The existence of different types of resonant states must be reflected in the existence of different types of S-matrix poles. Consequently it is important to find a method to identify simultaneously all the S-matrix poles. Let us consider the non-relativistic scattering of a charged particle by a central potential

$$\mathcal{V}(r) = gV_n(r) + V_{bar}(r), \tag{1}$$

where the short range complex nuclear potential V_n of strength $g \in \mathbf{C}$ has a square or a Woods-Saxon form-factor, and V_{bar} is a potential barrier. The dimensionless variable r/R will be used instead of r. For the sake of simplicity the notation r, k, g and c will be used for the dimensionless variables r/R, kR, $(\hbar^2/2MR^2)g$ and cR (where c is the Coulomb parameter $c = Z_1 Z_2 e^2 M/\hbar^2$). The S-matrix poles are the solutions $k = k_l(g)$ of the equation

$$\mathcal{F}_{l+}(g,k) = 0, \tag{2}$$

where $\mathcal{F}_{l+}(g,k)$ is the Jost function [1], l is the orbital angular momentum, k is the wave number and g is the potential strength, provided that $\mathcal{F}_{l-}(g,k) \neq 0$. The pole function $k = k_l(g)$ is a multiple-valued function defined on the complex g-plane. The S-matrix poles distribution in the k-plane as a function of the potential strength g has been extensively studied [2], [3], [4], [5], [6], [7] by using the *pole trajectory method*: a particular path in the complex g-plane is chosen and the corresponding trajectory of the S-matrix poles in the k-plane is determined. The pole trajectory method suffers from a poor treatment of the multiformity of the function $k = k_l(g)$: the method does not provide all S-matrix poles, some important S-matrix poles being lost, and one can never be sure that the same pole is followed.

The global method [1] involves the construction of the Riemann surface $R_g^{(l)}$ over the g-plane on which the pole function $k = k^{(l)}(g)$ is single valued and analytic. This implies the division of the Riemann surface $R_g^{(l)}$ into sheets $\Sigma_n^{(l)}$ and the construction of the Riemann sheets images $\Sigma_n^{'(l)}$ in the k-plane. If g takes values on a given Riemann sheet $\Sigma_n^{(l)}$ the pole $k = k^{(l)}(g)$ belongs to the Riemann sheet image $\Sigma_n^{\prime(l)}$ in the k-plane. The number n that labels the Riemann sheet $\Sigma_n^{(l)}$ and the Riemann sheet image $\Sigma_n^{\prime(l)}$ is used as a new quantum number for this pole and for the corresponding state (l,n). In this way, the sheet $\Sigma_n^{(l)}$ of the Riemann surface $R_g^{(l)}$ is associated to a given state with quantum numbers (l,n). It follows a novel insight into the intrinsic nature of the quantum states.

The Riemann surface approach to bound and resonant states, based on the global method for all S-matrix poles analysis, has several merits [2]: a) instead of analyzing an infinity of poles in the k-plane, the global method allows to analyze the single pole on each Riemann sheet image $\Sigma_n^{\prime(l)}$ in the k-plane. By analyzing each Riemann sheet image $\Sigma_n^{\prime(l)}$ no pole is lost; b) one associates to a given state (l,n) of the quantum system the sheet $\Sigma_n^{(l)}$ of the Riemann surface $R_g^{(l)}$. This approach allows us not only to study each state (l,n), but also to understand the transition from the state (l,n) to the state (l,m)

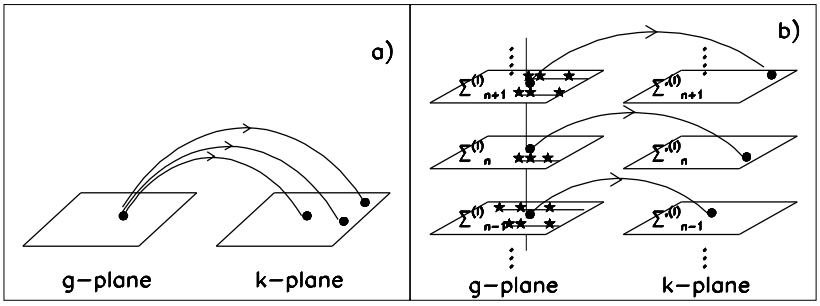

FIGURE 1. a) The multiple-valued function $k = k_l(g)$ defined on the complex g-plane and its distinct values in the k-plane. b) The Riemann surface over the complex g-plane. The branch-points indicated by $*$ and the branch-lines that join the sheets are shown. One can see that if g takes a value on a sheet $\Sigma_n^{(l)}$, then the function $k = k_l(g)$ takes only one value on the image of this sheet $\Sigma_n^{\prime(l)}$.

as a result of potential strength variation. Indeed, let us suppose that g describes a closed contour which starts from a point on the sheet $\Sigma_n^{(l)}$ and encloses the branch point joining the sheets $\Sigma_n^{(l)}$ and $\Sigma_m^{(l)}$. Then the pole passes from the sheet image $\Sigma_n^{\prime(l)}$ to the sheet image $\Sigma_m^{\prime(l)}$, i.e. the system makes a transition from the state (l,n) to the state (l,m), as a result of potential strength variation. Here the states (l,n) and (l,m) can be either bound or resonant states. As a result an unified treatment of the bound and resonant states is obtained; c) as g covers all the complex plane, for a given potential form factor $V(r)$ the well and the barrier with absorption or emission are treated simultaneously, which allows a smooth transition from one case to the other. If g follows a continuous path on a given sheet $\Sigma_n^{(l)}$, then the corresponding pole follows a continuous path in the k-plane sheet image $\Sigma_n^{\prime(l)}$; d) taking into account that on a Riemann sheet image $\Sigma_n^{\prime(l)}$ there is only one pole, the number n that labels the sheet $\Sigma_n^{(l)}$ and the sheet image $\Sigma_n^{\prime(l)}$ is used as a new quantum number, with topological meaning, for this pole and for the corresponding state (l,n); e) the global method for all S-matrix pole analysis is stable under the potential strength variation. Indeed, one cannot create or destroy S-matrix poles by varying the strength of the potential in the analyticity domain of the pole function $k = k^{(l)}(g)$. The poles can be created or destroyed only at the branch point $g = 0$. If g follows a path on a sheet $\Sigma_n^{(l)}$, the corresponding pole describes a trajectory remaining on the sheet image $\Sigma_n^{\prime(l)}$, provided that the path does not encircle a branch point and does not cross a small region containing the point $g = 0$.

The analysis of the Riemann surface has been done for three shapes of the potential: i) a square well followed by a square barrier; ii) a square or Woods-Saxon well with centrifugal barrier; iii) a square or Woods-Saxon well with Coulomb barrier. For all mentioned potential shapes a new class of poles, with unusual properties, has been identified.

The exotic resonant state poles and states have the following main properties:

i) instead of becoming bound or virtual state poles when the strength of the potential well increases to infinity, the exotic poles remain in some bound regions of the k-plane, in the neighborhood of some attractors (stable points). The number and position of these bounded regions depend on the shape and height of the barrier. They occur only if the absorptive potential strength $\mathcal{I}m\ g > 0$ belongs to a certain window $(t_1 \leq |\ \mathcal{I}m\ g\ | \leq t_2)$. Exotic poles exist only on some Riemann sheet images, depending on the shape of the potential barrier, as illustrated in fig. 2. One can see that for a rectangular well followed by a rectangular barrier there are exotic poles only for strong absorptive potentials. There is an infinite number of Riemann sheet images on which there are situated exotic poles. On each Riemann sheet image there is only one bounded region for the exotic resonant pole (fig 2a, 2b). In the case of a rectangular or Woods-Saxon well with centrifugal barrier there are exotic poles on a finite number of Riemann sheet images, the number of these sheet images increasing as the orbital angular momentum l increases. The exotic poles occur for either weak or strong absorptive potentials (fig. 2c, 2d). In the case of a rectangular or Woods-Saxon well with Coulomb plus centrifugal barrier there is an infinite

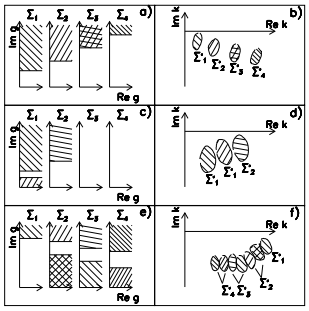

FIGURE 2. The first four sheets Σ_n, $n = 1, 2, 3, 4$ and the aggregates of their k-plane images Σ'_n for the three shapes of the potential. In fig. 2a and 2b the sheets and the aggregate of the sheet images for a square well followed by a square barrier are given; in fig. 2c and 2d the sheets and the aggregate of the sheet images for the potential made of a square or Woods-Saxon well with centrifugal barrier are given; in fig. 2e and 2f the sheets and the aggregate of the sheet images for the potential made of a square or Woods-Saxon well with Coulomb barrier are given.

number of Riemann sheet images where the exotic poles are situated, and the exotic poles occur for either strong or week absorption (fig. 2e, 2f);

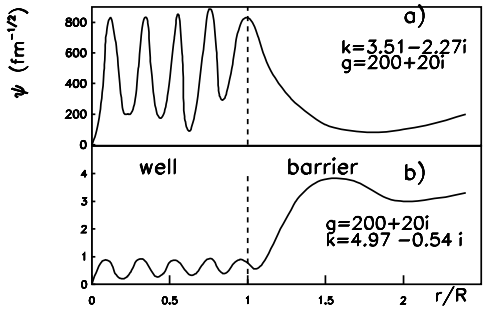

FIGURE 3. The moduli of the wave functions of an usual resonant state (a) and of an ERS (b) for the potential made of a square well followed by a square barrier with equal radii. The values of the potential well depth g and the corresponding poles in the k-plane are given.

ii) the wave functions of the exotic resonant states are localized in the region of the barrier, rather than in the region of the well; the wave functions of the exotic resonant states that correspond to poles situated in the neighborhood of the stable points are almost completely confined to the region of the barrier; the localization of the wave function in the case of potential (1) having the shape of a rectangular well followed by a rectangular barrier is illustrated in fig. 3;

iii) the resonant levels for the rectangular central potential exhibit a local degeneracy with respect to the orbital angular momentum l. Let $g_{j,i}^{(l)}$ $(j = 1, 2, \ldots)$ be the set of potential well strengths for which there is a resonant state of angular momentum l corresponding to a pole k situated at an attractor $\mathcal{K}_i^{(l)} \in \Sigma_i'^{(l)}$. In [2] it was demonstrated that there are three sheets belonging to three distinct Riemann surfaces $R_g^{(l)}$, $R_g^{(l-1)}$ and $R_g^{(l+1)}$ that are joined at a given value of the potential strength from the set $g_{j,i}^{(l)}$. A careful analysis of the Riemann surfaces shows that at $g_{j,i}^{(l)}$, for which there is a pole at the attractor $\mathcal{K}_i^{(l)} \in \Sigma_i'^{(l)}$, the sheets $\Sigma_q^{(l-1)}$, $\Sigma_i^{(l)}$ and $\Sigma_q^{(l+1)}$ $(l > 1)$, where $q = j + i - 1$ for even l and $q = j + i$ for odd l, are joined. In [1] the effect of the potential well diffuseness on the pole positions was studied. It was shown that the attractor $\mathcal{K}_i^{(l)}$ is slightly shifted and that the poles in the waves $(l-1)$ and $(l+1)$ are shifted too, provided that the diffuseness is small. Consequently, for a diffuse edge well

at $g_{j,i}^{(l)}$, for which there is a pole at the attractor $\mathcal{K}_i^{(l)} \in \Sigma_i^{\prime(l)}$, the levels $(l-1,q)$, (l,i) and $(l+1,q)$, where $q = j+i-1$ for even l and $q = j+i$ for odd l, are rather quasi-degenerate than degenerate.

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Uniform asymptotic approximation of 3-D Coulomb scattering wave function

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Abstract. The uniform asymptotic approximation for large η of the exact 3-D Coulomb scattering wave function Ψ_c is obtained. This approximation is deduced by using the closed form expressions of Ψ_c and of its components $\Psi_{\rm cdpw}$ and $\Psi_{\rm cdow}$ in terms of the Coulomb wave functions and their derivatives, obtained in the present Letter.

1. Introduction

The regular solution $\Psi_c(\mathbf{r})$ of the Schrödinger equation which describes the scattering by a pure Coulomb potential acting between two charged particles has been obtained by Gordon [1]. $\Psi_c(\mathbf{r})$ depends on two parabolic variables, $\xi = r - z$ and $\zeta = r + z$, where z is the axis along the incident beam, and on the Sommerfeld parameter $\eta = Z_1 Z_2 e^2 m/\hbar^2 k$. The usual separation of Ψ_c into two irregular solutions $\Psi_c = \Psi_i + \Psi_s$ is asymptotically satisfactory; from Ψ_s for $r \to \infty$ the scattering amplitude is usually derived [2]. However, in the non-asymptotic region neither Ψ_i, nor Ψ_s has a clear physical meaning [2]. The Gordon's solution is an important result in quantum mechanics. However, the form of the solution is not sufficiently simple for practical applications (e.g. Coulomb excitation [3]) and it is not suitable for the derivation of a 3-D WKB approximation or an uniform asymptotic approximation for large η of $\Psi_c(\mathbf{r})$.

It is well known that even for problems which have been solved exactly it often happens that only the asymptotic approximation of the solution is sufficiently simple to be useful in practical applications. Moreover, the asymptotic approximation stresses the functional dependence of the solution on the parameters. The most used approximation is the 3-D WKB approximation. However this approximation fails at the caustic, because the amplitude function becomes infinite there. In order to overcome the mentioned difficulty of the 3-D WKB approximation an uniform asymptotic approximation, valid near and away from the caustic, is necessary.

As far as we know there is no uniform asymptotic approximation reported for the solutions of a 3-D Schrödinger equation with repulsive Coulomb potential. An asymptotic approximation of the solution of the 3-D Schrödinger equation has been derived and applied to the case of Coulomb scattering by Khudyakov [4](see also [5]). As stressed by Khudyakov himself, his asymptotic approximation is not uniform on the whole range of the variables, but is valid in the neighbourhood of the caustic and at a finite distance from the caustic. Indeed, the term which involves the derivative of the Airy function Ai' and which becomes prominent away from the caustic [6] is not present in the asymptotic approximation given in [4]. Rowe [7] attempted to develop a uniform asymptotic approximation for large k of Ψ_c starting with the Gordon's solution. However, in this case the turning point $\xi_{tp} = 4\eta/k - 2\mathrm{i}/k$ depends on k and tends to the real ξ axis for large k. Due to this, the Rowe's asymptotic approximation, valid for small ξ, fails in the neighbourhood of the turning point. Consequently, it cannot be continued to $\xi \to \infty$ [8]. In other words it is not uniform. The above mentioned approximations start with the 3-D Schrödinger wave functions, without using partial waves decomposition. Another approach to make uniform the asymptotic approximations of the 3-D Schrödinger solution is based on the decomposition in partial waves (see e.g. [9, 10, 11] and references cited there). However, no uniform asymptotic approximation in the case of pure Coulomb scattering was obtained. In Section 2 we derive a simple closed form expression of the 3-D pure Coulomb scattering wave function $\Psi_c(\mathbf{r})$ in terms of the regular Coulomb wave function of zero orbital angular momentum and its derivative. This allows us to split Ψ_c into a Coulomb-distorted plane wave function $\Psi_{\rm cdpw}$ and a Coulomb-distorted outgoing wave function $\Psi_{\rm cdow}$. Simple closed form expressions of $\Psi_{\rm cdpw}$ and $\Psi_{\rm cdow}$ in terms of incoming and outgoing Coulomb wave functions respectively of zero orbital angular momentum and their derivatives are obtained too.

The deduced expressions of Ψ_c, $\Psi_{\rm cdpw}$ and $\Psi_{\rm cdow}$ admit a physical meaning in both the asymptotic and non-asymptotic region. In Section 3 the uniform asymptotic approximation of Ψ_c for large η is obtained starting with the exact closed-form solution obtained in Section 2. The fact that this uniform asymptotic approximation is deduced from the exact closed-form solution gives the possibility to check the Ansatz introduced in the pioneering works of Kravtsov [12] and Ludwig [13] (see also [5, 6, 9, 10, 14]).

2. Closed form expressions for $\Psi_c(\mathbf{r})$, $\Psi_{\rm cdpw}(\mathbf{r})$ and $\Psi_{\rm cdow}(\mathbf{r})$

The Gordons's solution [1] $\Psi_c(\mathbf{r})$ is

$$\Psi_c(\mathbf{r}) = \mathrm{e}^{-\pi\eta/2}\ \Gamma(1+\mathrm{i}\eta)\ \mathrm{e}^{\frac{\mathrm{i}}{2}k(\zeta-\xi)}\ {}_1F_1(-\mathrm{i}\eta;1;-\mathrm{i}k\xi), \tag{1}$$

where ${}_1F_1$ is the confluent hypergeometric function [15]. In the following we deduce the expression of $\Psi_c(\mathbf{r})$ in terms of the regular Coulomb wave function $F_0(\eta,\frac{1}{2}k\xi)$ and its derivative. We use the relations 1.9.2 and 2.4.9 of [15]

$$ {}_1F_1(a;1;x) = \mathrm{e}^{\frac{x}{2}}\ x^{-\frac{1}{2}}\ M_{\frac{1}{2}-a,0}(x), \tag{2}$$

$$ M_{b+\frac{1}{2},0}(x) = -\frac{1}{2}\sqrt{x}\ M_{b,\frac{1}{2}}(x) + \sqrt{x}\ M'_{b,\frac{1}{2}}(x), \tag{3}$$

and the relation

$$M_{\mathrm{i}\eta,\frac{1}{2}}(2i\rho) = 2\ \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(1-\mathrm{i}\eta)}\ |\Gamma(1+\mathrm{i}\eta)|^{-1}\ F_0(\eta,\rho) \tag{4}$$

(see [16]). Here $M_{p,q}(z)$ is the Whittaker function [15] and $F_0(\eta,\rho)$ is the regular Coulomb wave function of zero orbital angular momentum [16]. Taking $x=\mathrm{i}k\xi$, $a=b=-\mathrm{i}\eta$ and $\rho=\frac{1}{2}k\xi$ in the above relations the following expression of Ψ_c is obtained from Eq. (1)

$$\Psi_c(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\sigma_0}\ \mathrm{e}^{\mathrm{i}\frac{k}{2}\zeta}\ \left[F'_0(\eta,\tfrac{1}{2}k\xi) - \mathrm{i}F_0(\eta,\tfrac{1}{2}k\xi)\right], \tag{5}$$

where $\sigma_0 = \arg\Gamma(1+\mathrm{i}\eta)$ is the Coulomb phase shift for the orbital angular momentum $l=0$. The following comments on the closed-form expression of the solution Ψ_c given by Eq. (5) can be done: *i)* the solution is expressed in terms of the well-known regular Coulomb wave function of zero orbital angular momentum and its derivative; *ii)* the Gordon's solution (1) is expressed in terms of the complex valued function ${}_1F_1(-\mathrm{i}\eta;1;-\mathrm{i}k\xi)$, while the solution Ψ_c given by Eq. (5) is expressed in terms of two real valued functions F_0 and F'_0; *iii)* the solution is sufficiently simple to allow the derivation of the 3-D WKB and 3-D uniform asymptotic approximation for large η, as it will be shown in Section 3.

Let us compare the expression of $\Psi_c(\mathbf{r})$ obtained in the present paper to the well-known partial waves decomposition of $\Psi_c(\mathbf{r})$ [17]

$$\Psi_c(\mathbf{r}) = \frac{1}{kr}\sum_{l=0}^{\infty}(2l+1)\ \mathrm{i}^l\ \mathrm{e}^{\mathrm{i}\sigma_l}\ F_l(\eta,kr)\ P_l(\cos\theta), \tag{6}$$

where $\sigma_l = \arg\Gamma(l+1+\mathrm{i}\eta)$ is the Coulomb phase shift. Although the wave function Ψ_c in Eq. (5) is given in parabolic variables, it contains the Coulomb wave function F_0, which is characteristic to the partial wave $l=0$, but the original variable kr is replaced by $\frac{1}{2}k\xi$.

In the following we split the regular solution $\Psi_c(\mathbf{r})$ into two irregular solutions, representing the incident and scattered parts of Ψ_c. In order to do this we express F_0 in terms of the outgoing and incoming Coulomb wave functions $H_0^{(\pm)} = G_0 \pm \mathrm{i}F_0$ [16] via the relation $F_0 = -\frac{\mathrm{i}}{2}\left(H_0^{(+)} - H_0^{(-)}\right)$. It follows $\Psi_c = \Psi_{\rm cdpw} + \Psi_{\rm cdow}$, where

$$\Psi_{\rm cdpw}(\mathbf{r}) = -\frac{1}{2\mathrm{i}}\ \mathrm{e}^{\mathrm{i}\sigma_0}\ \mathrm{e}^{\mathrm{i}\frac{k}{2}\zeta}\left[H_0^{(-)\prime}(\eta,\tfrac{1}{2}k\xi) - \mathrm{i}H_0^{(-)}(\eta,\tfrac{1}{2}k\xi)\right], \tag{7}$$

$$\Psi_{\rm cdow}(\mathbf{r}) = \frac{1}{2\mathrm{i}}\ \mathrm{e}^{\mathrm{i}\sigma_0}\ \mathrm{e}^{\mathrm{i}\frac{k}{2}\zeta}\left[H_0^{(+)\prime}(\eta,\tfrac{1}{2}k\xi) - \mathrm{i}H_0^{(+)}(\eta,\tfrac{1}{2}k\xi)\right]. \tag{8}$$

By introducing in Eqs. (7) and (8) the asymptotic expansions for large ρ of $H_0^{(\pm)}(\eta,\rho)$ and of their derivatives [18]

$$H_0^{(\pm)}(\eta,\rho) \underset{\rho\to\infty}{\sim} \left[1+\frac{\eta}{2\rho}\pm\frac{\mathrm{i}\eta^2}{2\rho}\right]\mathrm{e}^{\pm\mathrm{i}(\rho-\eta\log 2\rho+\sigma_0)}, \tag{9}$$

$$H_0^{(\pm)\prime}(\eta,\rho) \underset{\rho\to\infty}{\sim} \left[-\frac{\eta^2}{2\rho}\pm\mathrm{i}(1-\frac{\eta}{2\rho})\right]\mathrm{e}^{\pm\mathrm{i}(\rho-\eta\log 2\rho+\sigma_0)}, \tag{10}$$

the following asymptotic expansions for large r, $(\theta\neq 0)$, are obtained

$$\Psi_{\mathrm{cdpw}}(\mathbf{r}) \underset{r\to\infty}{\sim} \left(1+\frac{\eta^2}{\mathrm{i}k\xi}\right)\mathrm{e}^{\mathrm{i}(kz+\eta\log k\xi)}, \tag{11}$$

$$\Psi_{\mathrm{cdow}}(\mathbf{r}) \underset{r\to\infty}{\sim} -\frac{\eta}{k\xi}\,\frac{\Gamma(1+\mathrm{i}\eta)}{\Gamma(1-\mathrm{i}\eta)}\,\mathrm{e}^{\mathrm{i}(kr-\eta\log k\xi)}. \tag{12}$$

One can check that Ψ_{cdpw} and Ψ_{cdow} are exact solutions of the 3-D Schrödinger equation with a Coulomb potential. According to Eqs. (11) and (12) as $r\to\infty$ they behave as a distorted plane and as a distorted outgoing wave, respectively. Consequently Ψ_{cdpw} will be called the Coulomb-distorted plane wave function and Ψ_{cdow} will be called the Coulomb-distorted outgoing wave function. In the absence of the Coulomb potential ($\eta=0$) from Eqs. (7) and (8) we obtain $\Psi_{\mathrm{cdpw}}=\mathrm{e}^{\mathrm{i}kz}$ and $\Psi_{\mathrm{cdow}}=0$, so that $\Psi_c=\mathrm{e}^{\mathrm{i}kz}$, as expected. We remark that both the incident and outgoing scattered waves are distorted by logarithmic phase factors. This is a direct consequence of the long range nature of the Coulomb potential. From Eq. (12) the Coulomb scattering amplitude can be deduced:

$$f_c(\theta) = -\frac{\eta}{2k\sin^2(\theta/2)}\,\frac{\Gamma(1+\mathrm{i}\eta)}{\Gamma(1-\mathrm{i}\eta)}\,\mathrm{e}^{-\mathrm{i}\eta\log\sin^2(\theta/2)}. \tag{13}$$

3. Uniform asymptotic approximation for large η of $\Psi_c(\mathbf{r})$

The above obtaned closed-form expression (5) of the solution $\Psi_c(\mathbf{r})$ allows us to reduce the problem of the 3-D WKB approximation of $\Psi_c(\mathbf{r})$ to the simpler problem of the 1-D WKB approximation. Indeed, according to Eq. (5), $\Psi_c(\mathbf{r})$ is expressed as a product of the exponential function $\mathrm{e}^{\frac{1}{2}k\zeta}$ and a linear combination of the Coulomb wave function $F_0(\eta,\frac{1}{2}k\xi)$ and its derivative $F_0'(\eta,\frac{1}{2}k\xi)$, which are functions of one variable $\rho=\frac{1}{2}k\xi$. Moreover, the WKB approximation of $F_0(\eta,\rho)$ and $F_0'(\eta,\rho)$ are well known [16]. The following WKB approximation of the solution $\Psi_c(\mathbf{r})$ is obtained

$$\Psi_c \sim \frac{1}{2}\,W^{-1/4}\,\mathrm{e}^{\mathrm{i}\sigma_0}\,[(W^{1/2}-1)\,\mathrm{e}^{\mathrm{i}S_1}+(W^{1/2}+1)\,\mathrm{e}^{\mathrm{i}S_2}], \tag{14}$$

where $W=1-\frac{4\eta}{k\xi}$ and $S_{1,2}=\frac{1}{2}k\left[\zeta\pm\int_{4\eta/k}^{\xi}\left(1-\frac{4\eta}{k\xi}\right)^{1/2}d\xi\right]$. This result can also be obtained from the Schrödinger equation (see, for example, [1] and [19]). From Eq. (14) one can see that the WKB approximation of the 3-D Coulomb scattering wave function $\Psi_c(\mathbf{r})$ fails at the caustic, because the amplitude function becomes infinite there. The caustic is the paraboloid defined by the equation $\xi=\frac{4\eta}{k}$ and separates the classically forbidden region $\left(0<\xi<\frac{4\eta}{k}\right)$ from the allowed region $\left(\xi>\frac{4\eta}{k}\right)$.

In order to overcome the mentioned difficulty of the WKB approximation, an uniform asymptotic approximation, valid near and away from the caustic, is necessary. The uniform asymptotic approximation reduces to the WKB approximation away from the caustic and remains finite at the caustic.

By using the exact closed-form (5) of the wave function Ψ_c describing the above derived Coulomb scattering we obtain an uniform asymptotic approximation for large η, whose main advantage is the possibility to describe the wave function over the entire space of the variables.

In order to obtain the uniform asymptotic approximation of $\Psi_c(\mathbf{r})$ for large η the asymptotic approximations of $F_0(\eta,\rho)$ and $F_0'(\eta,\rho)$ for large η and the unrestricted values of ρ will be used. $F_0(\eta,\rho)$

satisfies the differential equation

$$\frac{d^2y}{d\rho^2} + \left[1 - \frac{2\eta}{\rho}\right] y = 0, \tag{15}$$

which has a turning point at $\rho = 2\eta$. To the right of this turning point all solutions of Eq. (15) are oscillatory, while to the left of this point the solutions show a monotonic behaviour. In the neighbourhood of $\rho = 2\eta$ there is a transition region in which every solution changes from a monotonic to an oscillatory behaviour. Uniform asymptotic approximation for large η of $F_0(\eta,\rho)$ and $F_0'(\eta,\rho)$ in terms of Airy function holds uniformly in the interval $0 < \rho < \infty$ [20] and [21] (see also [8]).

With the notation

$$\frac{2}{3}[-\phi(x)]^{3/2} = \begin{cases} -\mathrm{i}f(x)\ , & \text{for } x > 0, \\ -\chi(x) + \frac{\pi}{2}, & \text{for } -1 < x \leq 0, \end{cases}$$

where

$$f(x) = [x(1+x)]^{1/2} - \log[(1+x)^{1/2} + x^{1/2}], \text{ for } x > 0, \tag{16}$$

$$\chi(x) = [-x(1+x)]^{1/2} - \frac{1}{2}\cos^{-1}(2x+1) + \frac{\pi}{2}, \text{ for } -1 < x \leq 0, \tag{17}$$

and $x = \dfrac{\rho - 2\eta}{2\eta}$, the asymptotic approximations for large η of $F_0(\eta,\rho)$ and $F_0'(\eta,\rho)$ in terms of the Airy function, valid for $0 < \rho < \infty$, are

$$F_0(\eta,\rho) \sim \pi^{1/2} \quad (2\eta)^{1/6}\ q(\rho), \tag{18}$$

$$F_0'(\eta,\rho) \sim -\pi^{1/2} \quad (2\eta)^{-1/6}\ \tilde{q}(\rho). \tag{19}$$

In Eqs. (18), and (19) the functions q and $\tilde{q}$ are given by

$$q(\rho) = [\phi'(x)]^{-1/2}\ \mathrm{Ai}[-(2\eta)^{2/3}\phi(x)], \tag{20}$$

$$\tilde{q}(\rho) = [\phi'(x)]^{1/2}\ \mathrm{Ai}'[-(2\eta)^{2/3}\phi(x)]. \tag{21}$$

The uniform asymptotic approximation of $\Psi_c(\mathbf{r})$ for large η is obtained by introducing in Eq. (5) the above asymptotic approximations of $F_0(\eta, k\xi/2)$ and $F_0'(\eta, k\xi/2)$ for large η and $0 < \xi < \infty$. For both the classically allowed region $\xi \geq 4\eta/k$ and classically forbidden region $0 < \xi \leq 4\eta/k$ one obtains

$$\Psi_c \sim -\mathrm{i}\pi^{1/2}(2\eta)^{1/6}\ \mathrm{e}^{\mathrm{i}\sigma_0}\ \mathrm{e}^{\mathrm{i}\frac{k}{2}\zeta}\left\{q(\tfrac{1}{2}k\xi) - \mathrm{i}(2\eta)^{-1/3}\ \tilde{q}(\tfrac{1}{2}k\xi)\right\}, \tag{22}$$

where q, and $\tilde{q}$ are given by Eqs. (20) and (21).

Taking into account the expressions (20) and (21) for the coefficients q and $\tilde{q}$ one can see that in both the classically allowed region ($\xi \geq 4\eta/k$) and the classically forbidden region ($0 < \xi \leq 4\eta/k$) the uniform asymptotic approximation of $\Psi_c(\mathbf{r})$ for large η is expressed in terms of the Airy function Ai and its derivative (see Eq. (22)). From Eq. (22) it follows that the wave function Ψ_c changes from an oscillatory behaviour on one side of the caustic (classically allowed region) to an exponential behaviour on the other side of the caustic (classically forbidden region).

Taking $\xi \to \infty$ in Eq. (22) we obtain again the Rutherford amplitude given by Eq. (13).

The uniform asymptotic approximation of the 3-D Coulomb scattering wave function $\Psi_c(\mathbf{r})$ obtained in the present paper was deduced from the exact closed-form solution (5), which is expressed in terms of functions of one variable $F_0(\eta, \frac{1}{2}k\xi)$, $F_0'(\eta, \frac{1}{2}k\xi)$ and $e^{\frac{i}{2}k\zeta}$. In this way the problem of uniform asymptotic approximation of the function of two variables $\Psi_c(\mathbf{r})$ was reduced to the uniform asymptotic approximation of the functions $F_0(\eta, \frac{1}{2}k\xi)$ and $F_0'(\eta, \frac{1}{2}k\xi)$, which are functions of one variable $\frac{1}{2}k\xi$.

In principle, the uniform asymptotic approximation of the 3-D Schrödinger equation solution is derived from an initial guess ("Ansatz") as to the form of the solution. Catastrophe theory paved the way for a general theory of uniform asymptotics based on standard integrals [22]. However, for a given problem, there is no general method to determine the appropriate "Ansatz" a priori. An accurate guess as to the form of the solution is crucial in order to avoid the occurrence of singularities in some of its initially undetermined coefficients [6]. Kravtsov [6, 12] and Ludwig [13] have used in the case of a

smooth convex caustic an Ansatz, which should be checked because the conditions of its applicability, given by Groshev and Kravtsov [6, 23], are necessary, but not sufficient.

Now consider the Ansatz in the theory of wave propagation. The propagation of the wave $u(\mathbf{r})$ with wave number k in a non-homogeneous medium of refractive index $n(\mathbf{r})$ is described by the reduced wave equation

$$\Delta u + k^2\ n^2(\mathbf{r})\ u = 0. \tag{23}$$

In order to obtain an asymptotic approximation for large wave number k of the solution of this equation, Kravtsov [6, 12] and Ludwig [13] have used in the case of a smooth convex caustic the following Ansatz

$$u(\mathbf{r}) \ = \ \mathrm{e}^{\mathrm{i}k\phi(\mathbf{r})} \left[\alpha(\mathbf{r})\ \mathrm{Ai}\left(\omega(\mathbf{r})\right) + \frac{1}{\mathrm{i}k}\beta(\mathbf{r})\ \mathrm{Ai}'\left(\omega(\mathbf{r})\right)\right], \tag{24}$$

where the four functions $\alpha(\mathbf{r})$, $\beta(\mathbf{r})$, $\phi(\mathbf{r})$ and $\omega(\mathbf{r})$ are determined by introducing this form into the reduced wave equation. Taking into account that, starting with the exact solution, we have obtained an uniform asymptotic approximation valid at a smooth convex caustic (paraboloid), we are able to check the Ansatz (24) in the case of Coulomb scattering. By comparing the uniform asymptotic approximation of Ψ_c for large η given by Eq. (22) to the Ansatz (24) one can see that the mentioned Ansatz is valid in both the classically forbidden region ($0 < \xi \leq 4\eta/k$) as well as in the classically allowed region ($\xi \geq 4\eta/k$).

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Performance metrics for multiobjective optimization evolutionary algorithms

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Abstract. In the recent past many algorithms for multiobjective optimization have been proposed. To evaluate performances of these algorithms some measures of performances are needed. Many metrics of algorithms performances have been proposed. The existing performance metrics are briefly reviewed. Two metrics computing the convergence towards the Pareto front and the solution diversity on the Pareto front are proposed.

Keywords: evolutionary algorithms, multiobjective optimization, Pareto front, performance metrics

1. Introduction

In the last years many multiobjective optimizations algorithms (MOA) have been proposed. To compute the performances of these algorithms some measures of performance were also introduced. Most of them are applied to the final nondominated set. It is now established that more than one metrics are necessary to evaluate the performances of the multiobjective evolutionary algorithms. Zitzler [5] has recently shown that for an M-objective optimization problem, at least M performance metrics must be used.

According to Deb [1] the existing performance metrics can be classified into three classes: metrics for convergence, metrics for diversity and metrics for both convergence and diversity.

Some of more recent and important metrics of performance are reviewed in the next section. Two new metrics one for convergence and one for diversity are introduced in Section 3.

2. Performance metrics: a review

Here we classify the here measures for evolutionary algorithms performances in two major classes:

- convergence metrics – evaluate how far from the true Pareto front solutions obtained in final population are;
- diversity metrics – evaluate scatter of solutions in the final population on the Pareto front.

Each of them is described in detail bellow.

2.1. Metrics for convergence.

Many metrics for measuring the convergence of a set of nondominated solutions towards the Pareto front have been proposed. Almost all of these metrics were constructed in order to directly compare two sets of nondominated solutions. There are also approaches which compare a set of nondominated solutions with a set of Pareto optimal solutions if the true Pareto front is known.

In what follows we review some existing metrics for convergence.

2.1.1. *Metric S.* The S metric has been introduced by Zitzler in [4] and improved in [5]. The S metric measures how much of the objective space is dominated by a given nondominated set A.

Definition 1 (Size of the dominated space)**.** *Let X be set of decision vectors for the considered problem and let $A = \{x_1, x_2, \ldots, x_t\} \subseteq X$ be a set of t decision vectors. The function $S(A)$ gives the volume enclosed by the union of the polytopes $p_1, p_2, \ldots, p_t$, where each p_i is formed by the intersection of the following hyperplanes arising out of x_i, along with the axes: for each axis in the objective space there exist a hyperplane perpendicular to the axis and passing through the point $(f_1(x_i), f_2(x_i), \ldots, f_k(x_i))$.*

Example 2. *In the two-dimensional case, each p_i represents a rectangle defined by the points (0,0) and $(f_1(x_i), f_2(x_i))$. An example for the two-dimensional case is presented in fig. 1.*

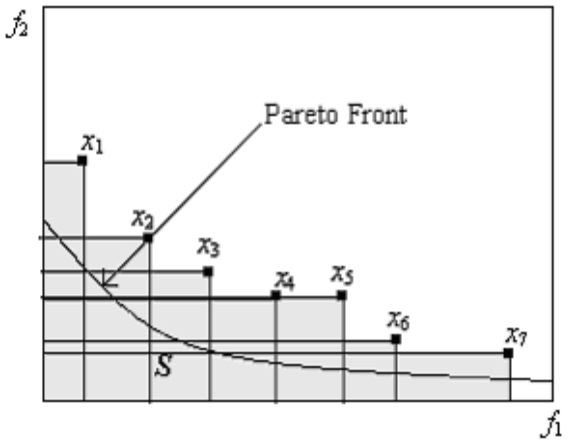

FIGURE 1. The metric$S$ for the case of two objective functions and 7 decision vectors $(x_1, x_2, \ldots, x_7)$ for a minimization problem.

2.1.2. Metric C The metric C, like the metric S, was introduced by Zitzler in [4] and improved in [5]. Using the metric C two sets of nondominated solutions can be compared to each other.

Definition 3 (Coverage of two sets)**.** *Let X be the set of decision vectors for the considered problem and let $A, B \subseteq X$ be two sets of decision vectors. The function C maps the ordered pair (A, B) into the interval [0,1]:*

$$C(A, B) = \frac{|\{b \in B \ / \ \exists a \in A : a \succeq b\}|}{|B|}.$$

Remark 4. *(i) The value $C(A, B) = 1$ means that all decision vectors in B are dominated by A.*
(ii) The value $C(A, B) = 0$ represent the situation when none of the points in B are dominated by A.
(iii) $C(A, B)$ is not necessary equal to $1 - C(B, A)$.

Example 5. *There are situations when the metric C cannot decide if an obtained front is better than the other. Let us suppose that front 1 correspond to a set A and front 2 to a set B.*

In fig. 2, the surface covered by the front 1 is equal to the surface covered by the front 2 but front 2 is closer to the Pareto optimal front than front 1. In this situation (and in other situations similar with this) the C metric is not applicable. To eliminate this shortcoming a new metric – D metric – was proposed.

Definition 6 (Coverage difference of two sets)**.** *Let $A, B \subseteq X$ be two sets of decision vectors. The size of the space dominated by A and not dominated by B (regarding the objective space) is denoted by $D(A, B)$ and is defined as:*

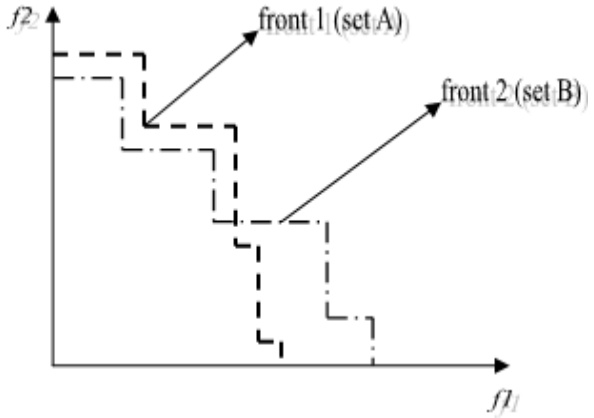

FIGURE 2. An example when the metric C can not decide between front 1 and front 2 (the surface covered by the front 1 is equal to the surface covered by the front 2).

$$D(A,B) = S(A+B) - S(B),$$

where $S(A)$ is defined above.

Example 7. *The metric D can be used to solve the inconvenience of Example 2. Consider the notation in fig. 3.*

By applying the metric D the followings equalities are obtained

$$S(A+B) = \alpha + \beta + \gamma;$$

$$S(A) = \alpha + \gamma;$$

$$S(B) = \alpha + \beta.$$

The metric D for this example is expressed below

$$D(A,B) = \gamma;$$

$$D(B,A) = \beta.$$

From

$$D(A,B) < D(B,A)$$

it follows that the front 2 dominates the front 1.

2.2. **Diversity metrics.** In this section the most frequently used metric for diversity is described.
2.2.1 A diversity metric In this section we consider a metric for diversity proposed by Deb in [2]. The obtained nondominated points at each generation are projected on a suitable hiperplan. The plan is divided into a number of small grids ($(M-1)$ dimensional boxes, M being the number of objectives). The diversity metric is defined according to on whether each grid contains an obtained nondominated point or not. The best possible result is obtained if all grids are represented with at least one point. If some grids are not represented by a nondominated point the diversity is poor.

Remark 8. *For greater number of objectives the value function will be difficult to define.*

3. TWO NEW METRICS FOR CONVERGENCE AND DIVERSITY

In this section two metrics - one for evaluate the convergence to the Pareto set and the other to determinate the spread of the solutions on the Pareto set are proposed.

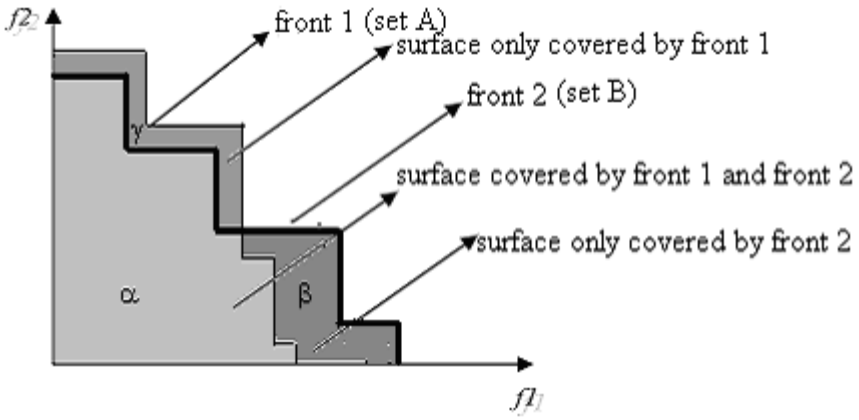

FIGURE 3. Example of difference between C metric and D metric for the considered fronts front 1 and front 2.

3.1. **New convergence metric.** Assume that the Pareto front is known. Let us denote by P a set of Pareto optimal solutions.

For each individual from the final population distance (Euclidian distance or other suitable distance) to all points of $P$is computed. The minimum distance is kept for each individual. The average of these distances represents the measure of convergence to the Pareto front.

3.2. **New diversity metric.** For each individual from the final population we consider the point from the set of Pareto optimal points P situated at the minimal distance. Several concepts of distance to a set may be considered. Here we consider $d(x,\, P)$ as being

$$d(x,\, P) = \min_{y \in P} d(x, y).$$

We called each such point from Pa *marked* point. The total number of different marked points from P over the size of P represents the diversity metric.

Remark 9. *These two metrics have a low computational cost. These metrics can be applied to high dimensional spaces.*

4. Conclusions

Many metrics have been proposed in the last years. Most of them calculate the convergence to an obtained set of solutions to the true Pareto front. The others measure the diversity of the obtained set of solutions on the Pareto front. We can not say that one metric is the best. Some of these metrics are preferred considering some aspects; the others, for the other aspects. Some of them are preferred to the others by considering the computation complexity. For different classes of problems different types of metrics can be preferred.

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Two generalizations of the uniform Gronwall lemma

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Abstract. We prove two generalizations of the uniform Gronwall lemma that are useful (at least) in obtaining upper bouns for various norms of solutions of several ODEs and PDEs.

1. Introduction

In proving the existence of solutions for ordinary differential equations (ODEs) or partial differential equations (PDEs), or in proving boundedness of solutions, the Gronwall Lemma is very often used. For instance, if the inequality

$$\frac{dy}{dt} \leq gy + h,$$

holds, where $g,\ h,\ y$ are locally integrable real functions defined on $[t_0, \infty)$, then the classical Gronwall lemma statesthat

$$y(t) \leq y(t_0) \exp\left(\int_{t_0}^{t} g(s)ds\right) + \int_{t_0}^{t} h(s) \exp\left(\int_{s}^{t} g(\theta)d\theta\right) ds, \qquad t \geq t_0. \tag{1}$$

If we try to prove the boundedness of the solution, the above inequality does not give good results for unbounded intervals of time, since the right-hand side has an exponential growth. In order to overcome this difficulty, in [1] a uniform Gronwall lemma is proved.

In order to make the ideas more clear we remind this Lemma in the form it appears in [2].

Lemma 1 (Uniform Gronwall Lemma of Foiaş and Prodi)**.** *Let g,h,y be three positive locally integrable functions on (t_0, ∞), with y' also locally integrable on (t_0, ∞) and that satisfy the inequalities*

$$\begin{aligned} \frac{dy}{dt} &\leq gy + h, \qquad \text{for} \qquad t \geq t_0, \\ \int_t^{t+r} g(s)ds &\leq a_1, \ \int_t^{t+r} h(s)ds \leq a_2, \ \int_t^{t+r} y(s)ds \leq a_3, \end{aligned}$$

where $r,\ a_1,\ a_2,\ a_3$ are positive constants. Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right) e^{a_1}, \qquad t \geq t_0.$$

Obviously, since r is fixed, the above inequality gives a result of uniform boundedness of the solution for any $t \geq t_0 + r$.

If we start from an inequality of the form

$$y(t) \leq y(s) + \int_s^t g(\tau)y(\tau)d\tau + \int_s^t h(\tau)d\tau,$$

we can not obtain the boundedness via the above form of the Uniform Gronwall Lemma.

The aim of our paper is to prove an Integral Uniform Gronwall Lemma to cover these situations. We also give a second generalization of the Uniform Gronwall Lemma.

2. Main results

Our first result is:

Lemma 2 (Integral Uniform Gronwall Lemma)**.** *Let g,h,y be three positive locally integrable functions on (t_0,∞), with y' also locally integrable on (t_0,∞) and that satisfy*

$$y(t) \leq y(s)+\int_s^t g(\tau)y(\tau)d\tau+\int_s^t h(\tau)d\tau, \qquad \text{for } s,\ t\geq t_0, \tag{2}$$

$$\int_t^{t+r} g(s)ds \leq a_1,\ \int_t^{t+r} h(s)ds\leq a_2,\ \int_t^{t+r} y(s)ds\leq a_3, \tag{3}$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r)\leq \left(\frac{a_3}{r}+a_2\right)(1+a_1e^{a_1}), \qquad t\geq t_0. \tag{4}$$

Proof. We write the inequality (2) in the form

$$y(t)\leq \int_s^t g(\tau)y(\tau)d\tau+\int_s^t\left[h(\theta)+\frac{1}{t-s}y(s)\right]d\theta.$$

By using an integral form of the Gronwall Lemma (e.g. [Wl]) we infere

$$\begin{aligned} y(t) \leq & \int_s^t\left[h(\theta)+\frac{1}{t-s}y(s)\right]d\theta+\\ & +\int_s^t\int_s^\tau\left(h(\alpha)+\frac{1}{t-s}y(s)\right)g(\tau)\exp\left(\int_\tau^t g(\theta)d\theta\right)d\alpha d\tau. \end{aligned}$$

By changing t by $t+r$, we have

$$y(t+r)\leq \int_s^{t+r}h(\theta)d\theta+y(s)+\int_s^{t+r}\int_s^\tau h(\alpha)g(\tau)\exp\left(\int_\tau^{t+r}g(\theta)d\theta\right)d\alpha d\tau+$$

$$+y(s)\int_s^{t+r}g(\tau)\exp\left(\int_\tau^{t+r}g(\theta)d\theta\right)d\tau.$$

Further, by taking $s\in[t,t+r]$, and by using the inequalities (3) we obtain

$$y(t+r)\leq a_2+y(s)+a_2a_1e^{a_1}+y(s)a_1e^{a_1}.$$

By integrating with respect to s between t and $t+r$ the preceding inequality, we get

$$ry(t+r)\leq ra_2(1+a_1e^{a_1})+(1+a_1e^{a_1})\int_t^{t+r}y(s)ds$$

and therefore

$$y(t+r)\leq a_2(1+a_1e^{a_1})+(1+a_1e^{a_1})\frac{a_3}{r},$$

hence (4). □

The second generalization of the Foiaş-Prodi Lemma is the following

Lemma 3. *Let g,h be two positive locally integrable functions on (t_0,∞), and $y:(t_0,\infty)\to\mathbb{R}$, such that $|y|$ and $|y'|$ are locally integrable. Suppose that*

$$\left|\frac{dy}{dt}\right| \leq g|y|+h, \qquad \text{for} \qquad t\geq t_0, \tag{5}$$

$$\int_t^{t+r}g(s)ds \leq a_1,\ \int_t^{t+r}h(s)ds\leq a_2,\ \int_t^{t+r}|y(s)|\,ds\leq a_3,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$|y(t+r)|\leq\left(\frac{a_3}{r}+a_2\right)(1+a_1e^{a_1}), \qquad t\geq t_0. \tag{6}$$

Proof. We have successively

$$||y(t)| - |y(s)|| \leq |y(t) - y(s)| \leq \left| \int_s^t y'(\tau) d\tau \right| \leq \int_s^t |y'(\tau)| \, d\tau.$$

By integrating (5) between s and t the inequality

$$\int_s^t |y'(\tau)| \, d\tau \leq \int_s^t g(\tau) \, |y(\tau)| \, d\tau + \int_s^t h(\tau) d\tau$$

is obtained, and thus the inequality

$$|y(t)| \leq |y(s)| + \int_s^t g(\tau) \, |y(\tau)| \, d\tau + \int_s^t h(\tau) d\tau$$

is obtained.

With the previous lemma, (6) follows. □

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The existence and fractal dimension of an inertial set for a convection problem

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Abstract. The problem we consider describes pattern formation in the Bénard convection. For this problem, in [7], [8], [5] the existence of the global attractor and that of an inertial manifold were proved. In the present paper we prove the existence of an inertial set and give an upper bound for its fractal dimension.

1. Introduction

In the qualitative theory of dissipative dynamical systems, the large time behavior asks for the study of global attractors. Since generally the global attractor is a complicated set in an infinite dimensional function space, the notion of inertial manifold was imposed [3]. This is a finite dimensional, Lipschitz manifold, positive invariant to the dynamical system, that exponentially attracts all the orbits. The positive invariance of the inertial manifold allows us to reduce the study of the infinite dimensional dynamical system to the study of the finite dimensional problem, called the inertial form.

In [2] the inertial set (also called exponential attractor) is defined as a subset (not necessarily a manifold) of the space phase that contains the global attractor, is positively invariant, has finite fractal dimension and exponentially attracts the orbits starting from a compact connected invariant set in the phase space. In the same work a theorem of existence of the inertial set is proved for a large class of problems.

It was shown that for many problems the fractal dimension of the inertial set is less than the dimension of the inertial manifold [2]. In addition, a theoretical frame for constructing a finite dynamical system that is a restriction of the given dynamical system obtained by means of the inertial set is settled.

In our paper we consider a problem following from a model of pattern formation in Bénard convection, studied by us in [5], [6], [7], [8]. In these papers we proved the existence of solutions for zero boundary conditions and for periodic boundary conditions. We also proved the existence of the global attractor, that of an inertial manifold and constructed a family of approximate inertial manifolds for it.

Like in [5], here we consider periodic boundary conditions. The existence of the inertial set is a straightforward consequence of a theorem from [2]. After checking its hypotheses for our problem, we give an explicit upper bound for the fractal dimension of this inertial set.

2. The formulation of the problem

Let us consider the equation:

$$\begin{aligned} \frac{\partial u}{\partial t}+\frac{1}{l^4}\Delta\Delta u+\frac{2}{l^2}\Delta u+2u+g(u) &= 0, \\ u &= u(t,x),\quad x\in\Omega \end{aligned} \tag{1}$$

where $\Omega=(-1,1)\times(-1,1)$, and $g(u)=u^3+\beta u^2-(R+1)u$. It models the pattern formation in Bénard convection [4], [10]. With (1) we associate the initial condition

$$u(x,0)=u_0(x),\qquad u_0\in L^2_{per}(\Omega) \tag{2}$$

and as boundary conditions we consider periodic boundary conditions.

Let $\mathcal{H}=L^2_{per}(\Omega)=\left\{u\in H^2(\Omega),\ \gamma_0 u|_{\Gamma_1}=\gamma_0 u|_{\Gamma_2},\ \Gamma_1,\Gamma_2 \text{ opposite edges of } \Omega\right\}$ with the usual scalar product and norm denoted $(\ ,\ )$ respectively $|\ \ |_0$. Let

$$\mathcal{V}=H^2_{per}(\Omega)=\left\{u\in H^2(\Omega),\ \gamma_j u|_{\Gamma_1}=(-1)^j\ \gamma_j u|_{\Gamma_2},\ \ \Gamma_1,\Gamma_2 \text{ opposite edges of } \Omega, j=\overline{0,2}\right\}$$

with the scalar product: $((u,v)) = \int_\Omega \Delta u \cdot \Delta v dx + \int_\Omega uv dx$ and the induced norm denoted $|\quad|_2$. In the above, γ_j are the trace operators of order j.

We have $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ the first embedding being compact. Let $Au = \Delta\Delta u + 2\Delta u + 2u$, $D(A) = H^4_{per}(\Omega)$.

Consider the space $D(A^{\frac{1}{2}})$ with the norm $\|u\| = \left|A^{\frac{1}{2}}u\right|_0$. Let us remark that

$$\min\{\frac{1}{3l^4}, \frac{1}{2}\}\,|u|_2^2 \leq \left|A^{\frac{1}{2}}u\right|_0^2 \leq \max\{\frac{2}{l^4}, 3\}\,|u|_2^2, \tag{3}$$

and thus, the norms $\|\quad\|$ and $|\quad|_2$ being equivalent, the spaces $\mathcal{V}$ and $D(A^{\frac{1}{2}})$ coincide.

With the same reasonings as those used in the paper [8], with minimal changes due to the use of the periodic space we prove the existence and uniqueness of the solution (1) (the Galerkin- Faedo method) (as we pointed out in [5]). More precisely we have the following result.

Proposition 1. *For every $u_0 \in \mathcal{H}$, the problem (1)-(2) has a unique solution u in $C(0,T;H^4(\Omega) \cap H^2_{per}(\Omega))$. The equation (1) is satisfied in the sense of the equality in $L^2(\Omega)$. The solution continuously depends on u_0.*

In the following we use the inequality

$$\int_\Omega u^{2j}dx \leq K_j\,|u|_0^{j+1}\,|u|_2^{j-1} \tag{4}$$

which relies upon the embedding $H^{\frac{j-1}{j}}(\Omega) \to L^{2j}(\Omega)$ and on the interpolation inequality [1]

$$[u]_{\frac{j-1}{j}} \leq |u|_0^{\frac{j+1}{2j}}\,|u|_2^{\frac{j-1}{2j}}.$$

With the same method as in [8], we prove the existence of an absorbing ball in $\mathcal{H}$, having radius $\rho_0 = 4\sqrt{C_1}$, where C_1 has the property $\frac{1}{2}u^4 - C_1 \leq g(u)u$. We can prove that $C_1 = \frac{8}{3}(\beta^2 + R + 1)^2$ is convenient.

We also prove as in [8], the existence of an absorbing ball in $\mathcal{V}$ of radius

$$\rho_1 = 4\sqrt{C_1} + \left[8C_1 + \rho_0^2 + \frac{4(R+1)^2\rho_0^2}{l^4}\right]^{\frac{1}{2}} e^{\delta}$$

with $\delta = 6 + 2l^4\rho_0^2\left(K_3\rho_0^2 + K_2\beta^2\right)$.

We aim to prove the existence of an inertial set for the above problem and use a form of the theorem proved in [2]. Let us settle the theoretical framework in which this theorem is proved.

3. Theoretical framework [2]

Let H be a separable Hilbert space and let $A : D(A) \subset H \to H$, be a linear self-adjoint positive operator, having a compact inverse. Consider the problem

$$\begin{aligned} \frac{du}{dt} + Au + R(u) &= 0 \\ u(0) &= u_0 \end{aligned} \tag{5}$$

where $R : D(A) \to H$ is locally Lipschitz from $D(A^{\frac{1}{2}})$ into H.

The problem (5) is supposed to have a unique solution $u(.,u_0) : \mathbb{R}^+ \to D(A)$ for any $u_0 \in H$ and, thus, a semigroup of operators $\{S(t)\}_{t\geq 0}$ may be defined. It is also assumed that the operators $S(t) : H \to D(A)$ are continuous for every $t \geq 0$ and that there is an absorbing ball for $\{S(t)\}_{t\geq 0}$ in H having radius ρ_0 and also an absorbing ball in V having radius ρ_1.

Let X be a set in $D(A^{\frac{1}{2}})$, that is compact in H, connected and positively invariant to $\{S(t)\}_{t\geq 0}$.

Denote by c_0 the Lipschitz constant of R on X, i.e.

$$|R(u) - R(v)|_0 \leq c_0\left|A^{\frac{1}{2}}(u-v)\right|_0$$

for every $u, v \in X$.

In this framework let us recall the definition of an inertial set.

A compact set $\mathcal{M} \subset H$ is an inertial set for $\{S(t)\}_{t\geq 0}$ if:
- the global attractor $\mathcal{A}$ is contained in $\mathcal{M}$;
- $\mathcal{M}$ is positively invariant to $\{S(t)\}_{t\geq 0}$;
- $\mathcal{M}$ has a finite fractal dimension;
- $\exists\, a_0,\ a_1 > 0$ such that for all $t \geq 0$, $dist(S(t)X,\ \mathcal{M}) \leq a_0 \exp(-a_1 t)$.

We use the following result which is a particular form ($\beta = \frac{1}{2}$, $\delta_* = \frac{e}{24}$ with the notations of [2]) of the theorem that asserts the existence of the inertial set.

Theorem 2. [2] *In the above hypotheses, the semi-dynamical system $\{S(t)\}_{t\geq 0}$ possesses an inertial set $\mathcal{M}$ with fractal dimension $d_F(\mathcal{M})$,*

$$d_F(\mathcal{M}) \leq \frac{2\ln 7}{\ln 6 - 1} N_0 + 1$$

where N_0 is such that

$$\lambda_{N_0+1} > 4c_0^2(3\ln 2 + 1),$$

where λ_{N_0+1} is the $N_0 + 1^{th}$ eigenvalue in the increasing sequence of eigenvalues of A.

4. The existence of the inertial set

The hypotheses on the linear operator A are obviously satisfied.

The fact that g is Lipschitz from $D(A^{\frac{1}{2}})$ into $\mathcal{H}$ was proved in [5], hence the existence of the inertial set is proved. However, here we perform the detailed computation in order to find the Lipschitz constant, c_0. We define the set $X = \{u \in \mathcal{H} \,|\, |u|_2 \leq \rho_1\}$. It is compact in $\mathcal{H}$ since the embedding of $\mathcal{V}$ in $\mathcal{H}$ is compact. Below we consider $u,\ v \in X$. We have

$$\begin{aligned} |g(u) - g(v)|_0^2 &= \int_\Omega \left[u^3 + \beta u^2 - (R+1)u - v^3 - \beta v^2 + (R+1)v\right]^2 dx \\ &\leq 3\int_\Omega (u-v)^2 \left(u^2 + uv + v^2\right)^2 dx + 3\beta^2 \int_\Omega (u-v)^2 (u+v)^2 dx + \\ &\quad + 3(R+1)^2 \int_\Omega (u-v)^2 dx. \end{aligned} \tag{6}$$

For the first integral, by using (4) for $j = 2$, we obtain

$$\begin{aligned} \int_\Omega (u-v)^2 \left(u^2 + uv + v^2\right)^2 dx &\leq \left[\int_\Omega (u-v)^4 dx\right]^{\frac{1}{2}} \left[\int_\Omega \left(u^2 + uv + v^2\right)^4 dx\right]^{\frac{1}{2}} \\ &\leq K_2^{\frac{1}{2}} |u-v|_0^{\frac{3}{2}} |u-v|_2^{\frac{1}{2}} \frac{9}{\sqrt{2}} \left[\int_\Omega \left(u^8 + v^8\right) dx\right]^{\frac{1}{2}}. \end{aligned}$$

Since $u \in X$, inequality (4) for $j = 4$ implies

$$\int_\Omega u^8 dx \leq K_4 |u|_0^5 |u|_2^3 \leq K_4 \rho_1^8,$$

and thus

$$\int_\Omega (u-v)^2 \left(u^2 + uv + v^2\right)^2 dx \leq 9 K_2^{\frac{1}{2}} K_4^{\frac{1}{2}} \rho_1^4 |u-v|_2^2.$$

For the second integral in (6), again with (4) for $j = 2$, we have

$$\begin{aligned}\int_\Omega (u-v)^2 (u+v)^2\, dx &\leq \left[\int_\Omega (u-v)^4\, dx\right]^{\frac{1}{2}} \left[\int_\Omega (u+v)^4\, dx\right]^{\frac{1}{2}} \\ &\leq K_2^{\frac{1}{2}} |u-v|_0^{\frac{3}{2}} |u-v|_2^{\frac{1}{2}} \left[8\int_\Omega \left(u^4+v^4\right) dx\right]^{\frac{1}{2}} \\ &\leq 4K_2\rho_1^2 |u-v|_2^2 .\end{aligned}$$

Finally, we obtain

$$|g(u)-g(v)|_0 \leq \left[27K_2^{\frac{1}{2}} K_4^{\frac{1}{2}} \rho_1^4 + 12K_2\rho_1^2\beta^2 + 3(R+1)^2\right]^{\frac{1}{2}} |u-v|_2 ,$$

and by means of inequality (3), we find that

$$c_0 = \left[\min\{\frac{1}{3l^4}, \frac{1}{2}\}\right]^{-\frac{1}{2}} \left[27K_2^{\frac{1}{2}} K_4^{\frac{1}{2}} \rho_1^4 + 12K_2\rho_1^2\beta^2 + 3(R+1)^2\right]^{\frac{1}{2}} .$$

The eigenvalues of the operator A are $\lambda_{j,k} = \left[\frac{\pi^2(j^2+k^2)}{l^2} - 1\right]^2 + 1$ and, with the help of a method used in [9] for the eigenvalues of the Laplace operator, we find that

$$\lambda_N \geq \frac{3\pi^2}{4l^4}\left(\frac{N}{4} - 1\right)^2 - 3,$$

where $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$ is the ordered sequence of eigenvalues of A.

In order to find an upper bound for the fractal dimension of the inertial set we impose the condition

$$\frac{3\pi^2}{4l^4}\left(\frac{N}{4} - 1\right)^2 - 3 \geq 4c_0^2(3\ln 2 + 1). \tag{7}$$

Let us define

$$N_0 = \left[\frac{8l^2}{\sqrt{3}\pi}\left(4c_0^2(3\ln 2+1)+3\right)^{\frac{1}{2}}\right] + 5,$$

where the square brackets stand for the integer part. This N_0 satisfies the condition (7) and the theorem above asserts that

$$d_F(\mathcal{M}) \leq \frac{2\ln 7}{\ln 6 - 1} N_0 + 1.$$

Thus, resuming all the results, we proved the existence of the inertial set for our problem and the following estimate for its fractal dimension:

$$d_F(\mathcal{M}) \leq \frac{2\ln 7}{\ln 6 - 1}\left\{\left[\frac{8l^2}{\sqrt{3}\pi}\left(4\frac{27K_2^{\frac{1}{2}} K_4^{\frac{1}{2}} \rho_1^4 + 12K_2\rho_1^2\beta^2 + 3(R+1)^2}{\min(\frac{1}{3l^4}, \frac{1}{2})}\ln 8e + 3\right)^{\frac{1}{2}}\right] + 5\right\} + 1,$$

where $\rho_0 = 4\sqrt{C_1}$, $\quad C_1 = \frac{8}{3}(\beta^2 + R + 1)^2$

$$\rho_1 = 4\sqrt{C_1} + \left[8C_1 + \rho_0^2 + \frac{4(R+1)^2\rho_0^2}{l^4}\right]^{\frac{1}{2}} e^\delta, \quad \delta = 6 + 2l^4\rho_0^2\left(K_3\rho_0^2 + K_2\beta^2\right).$$

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Unsaturated water flow through layerd media

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Abstract. A fast method is given to obtain an accurate solution of one dimension Richards' equations that models the water inflitration through porous media. The finite volume scheme with imposed flux continuity for the space discretization is used. One uses the backward differentiation formulae (BDF) for the time integration and one presents some numerical results.

Keywords: partial differential equations, method of lines, differential algebraic equation, discontinous coefficients, flux continuity

2000 Mathematics Subject Classification: 34A09, 35K15, 65H10, 65L80, 65M20, 76S05

1. Introduction

The basic equation to study the infiltration of water in unsaturated porous media is Richards' equation. This equation is derived from a combination of the equation of continuity and Darcy equation. This PDE equation covers both unsaturated and saturated flow regime. The Richards' equiation is of nonlinear parabolic type in the unsaturated domain flow or of linear elliptic type in the saturated domain. The Richards' equation can be cast in several forms, according to whether the pressure, water content or both are used as state variables. The only state variable that can be used in variable saturated/unsaturated flow is the pressure.

The integral form of the mass balance for an arbitrary domain V is given by

$$\partial_t \int_V \rho\theta \mathrm{d}x + \int_{\partial V} \rho \mathbf{v}\cdot\mathbf{n}\mathrm{d}s = 0, \tag{1}$$

where ρ stands for the water density, θ denotes the volumetric water content and $\mathbf{v}$ represents the Darcy velocity. The local form of (1) reads

$$\partial_t(\rho\theta) + \mathrm{div}\rho\mathbf{v} = 0. \tag{2}$$

In order to obtain a complete model of the water flow, in many applications the water content and the velocity are given as functions of the pressure head h

$$\begin{aligned} \mathbf{v} &= -K(h)\nabla(h+z)\,, \\ \theta &= \theta(h)\,. \end{aligned} \tag{3}$$

Here $K(h)$ represents the hydraulic conductivity and z denotes the co-ordinate along the vertical axis Oz oriented upward. If ρ is constant, from (2) and (3) one obtains the following $\theta - h$ form of the Richards' equation

$$\partial_t\theta + \mathrm{div}(-K\mathrm{grad}(h+z)) = 0. \tag{4}$$

On an interface separating two media, physical considerations require the continuity of the pressure head and the normal component of the velocity. Therefore, we have

$$\begin{aligned} h|_- &= h|_+\,, \\ \mathbf{v}\cdot\mathbf{n}|_- &= \mathbf{v}\cdot\mathbf{n}|_+\,. \end{aligned} \tag{5}$$

In this paper we present a numerical method to solve the equation (4). The method consists in two main steps: the space discretization and the time integration. For the space discretization of the Richards' equation we use the finite volume method (FVM). The net is defined such that an interface between

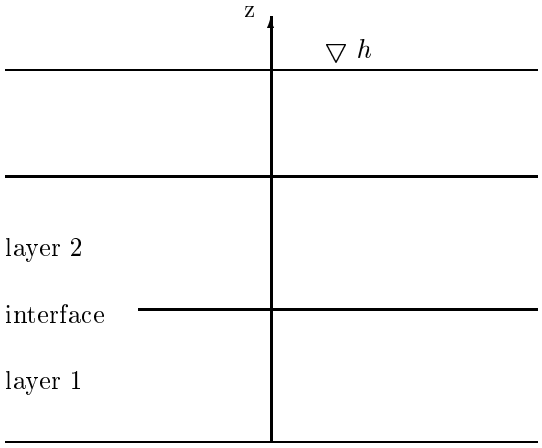


FIGURE 1. Layered soil.

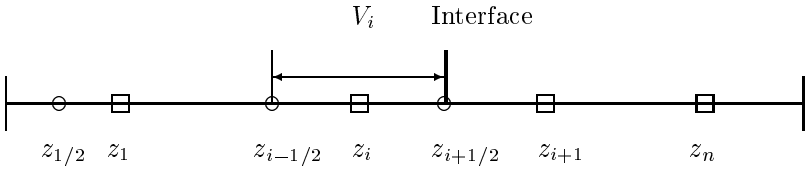


FIGURE 2. Control volumes.

two different layers does not intersect its elements. On the interface, the numerical flux is defined by a formula that reflects the continuity of the pressure head and the continuity of the normal component of the velocity. For the time integration, we use an implicit type backward method.

2. Space discretization

Consider a layered soil as in fig. 1. Define the control volumes by $V_i = (z_{i-1/2},\, z_{1+1/2})$ and denote by z_i the middle point of the control volume (fig. 2).

The net $\{z_{i-1/2}\}$ is defined such that any V_i is entirely included into a single layer. Therefore, one interface is located at some $z_{i-1/2}$. The integral form (1) for one V_i has the following expression

$$\partial_t \int_{z_{i-\frac{1}{2}}}^{z_{i+\frac{1}{2}}} \theta \mathrm{d}z + v_{i+1/2}(t) - v_{i-1/2}(t) = 0. \tag{6}$$

In this equation the integral is approximated by the middle point quadrature formula and the velocities are defined by the divided differences of pressure head

$$v_{i+1/2} \approx -K_{i+1/2} \frac{(h_{i+1} + z_{i+1}) - (h_i + z_i)}{z_{i+1} - z_i}. \tag{7}$$

The numerical hydraulic conductivty $K_{i+1/2}$ is defined as

$$\frac{1}{K_{i+1/2}} = \frac{1}{K_-^*}\frac{z^* - z_i}{z_{i+1} - z_i} + \frac{1}{K_+^*}\frac{z_{i+1} - z^*}{z_{i+1} - z_i}, \tag{8}$$

where z^* is a point on the interface, h^* is given by

$$h^* = h_{i+1}\frac{z^* - z_i}{z_{i+1} - z_i} + h_i\frac{z_{i+1} - z^*}{z_{i+1} - z_i},$$

and $K_-^*(K_+^*)$ denote the left (right) limit of the hydraulic conductivity.

We note that the numerical scheme that aproximates the space differential operator $\partial_z(K(h)\partial_z(h+z))$ is conservative and the numerical hydraulic conductivty is given by the formula (8) which represents the physical conditions (5). In the case of the linear elliptic equation with discontinuous coeficients it was shown [7] that the conservativeness is a necessary condition for a numerical scheme to be convergent. Also, for the linear case, a formula like (8) was obtained in the papers [1], [3].

We obtain the differential algebraic equations (DAE)

$$\begin{cases} \mathbf{E}\partial_t\boldsymbol{\theta} + \mathbf{A}(\mathbf{h})\mathbf{h} + \mathbf{g}(\mathbf{h}) &= 0, \\ \boldsymbol{\theta} &= \boldsymbol{\theta}(\mathbf{h}), \end{cases} \tag{9}$$

with $\mathbf{A}$ a threediagonal matrix, $\mathbf{g}$ vector and $\mathbf{E}$ diagonal matrix: $E_{i,i} = \mathrm{vol}(V_i)$.

3. Time Integration

In order to obtain the numerical solution of the DAE (9) we use an implicite backward differentiation formulae (BDF) for time discretization [2]. Let k be the order of the method, let $\{t_{n-k}, \cdots, t_n, t_{n+1}\}$ be a sequence of moments of time, let $\triangle t = t_{n+1} + t_n$ be the time step and let $\{\boldsymbol{\theta}^{n-k}, \cdots, \boldsymbol{\theta}^n\}$ and $\mathbf{h}^{n+1}$ be the corresponding values of the water content and of the pressure head respectively. Consider the *polynomial predictor* $\mathbf{q}^P(t)$ which interpolates the values of the water content at the time levels $\{t_{n-k}, \cdots, t_n\}$ given by

$$\mathbf{q}^P(t) = \sum_{i=0,k} b_i(t)\boldsymbol{\theta}^{n-i}, \tag{10}$$

where

$$b_i(t) = \prod_{j=0,k,j\neq i} \frac{t - t_{n-j}}{t_{n-i} - t_{n-j}}. \tag{11}$$

The *polynomial corrector* $\mathbf{q}^C$ is determined by the conditions

$$\begin{aligned} &\mathbf{q}^C(t_{n+1}) = \boldsymbol{\theta}^{n+1}, \\ &\mathbf{q}^C(t_{n+1} - l\triangle t) = \mathbf{q}^P(t_{n+1} - l\triangle t); \quad l = \overline{1,k}. \end{aligned} \tag{12}$$

Now, the pressure head $\mathbf{h}^{n+1}$ is obtained from the conditions that the time derivative of the polinomial corrector satisfies the DAE (9)

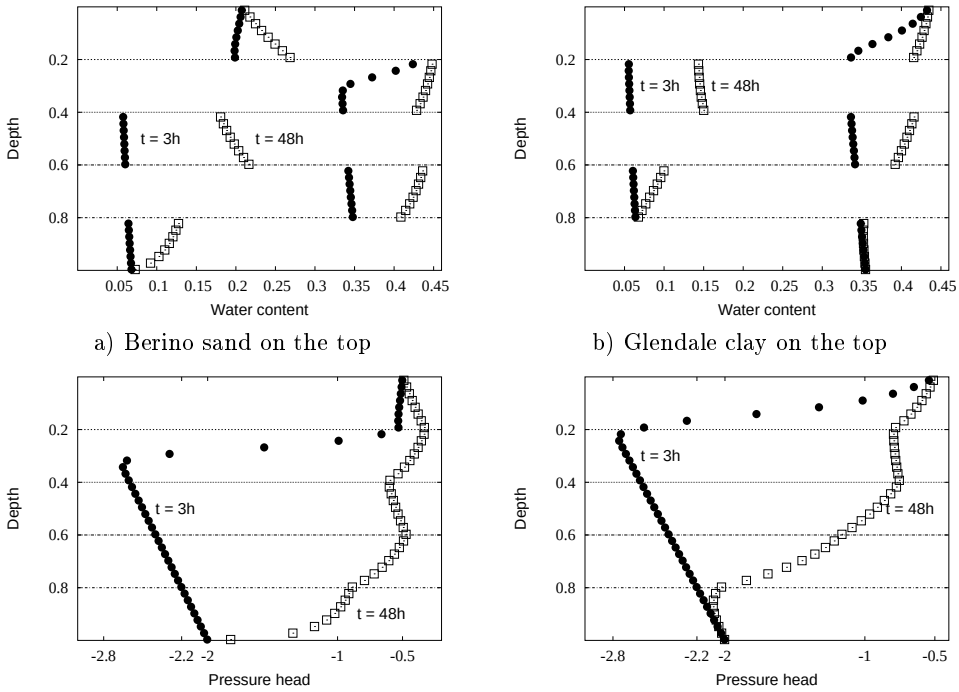
$$\mathbf{E}\left(\dot{\mathbf{q}}^P(t_{n+1}) + a\frac{\boldsymbol{\theta}(\mathbf{h}^{n+1}) - \mathbf{q}^P(t_{n+1})}{\triangle t}\right) + \mathbf{f}(\mathbf{h}^{n+1}) = 0, \tag{13}$$

where the constant a and $\mathbf{f}$ are given by

$$\begin{aligned} a &= 1 + 1/2 + \cdots + 1/k, \\ \mathbf{f}(\mathbf{h}) &= \mathbf{A}(\mathbf{h})\mathbf{h} + \mathbf{g}(\mathbf{h}). \end{aligned} \tag{14}$$

The nonlinear system (13) is solved iteratively. Let Ψ designate the l.h.s. of (13) and let $\mathbf{x}$ denote the unknown $\mathbf{h}^{n+1}$. The iteration has the form

$$\begin{aligned} \Phi^k\delta\mathbf{x}^k &= -\Psi^k, \\ \mathbf{x}^{k+1} &= \delta\mathbf{x}^k + \mathbf{x}^k. \end{aligned} \tag{15}$$



a) Berino sand on the top b) Glendale clay on the top

c) Berino sand on the top d) Glendale clay on the top

FIGURE 3. Water infiltration into layered soil. Pressure head is a continous function in the domain while water content has a jump across the interface.

The iterative matrix used here is given by

$$\Phi^k = \frac{a}{\triangle t}\mathbf{E}\nabla\boldsymbol{\theta}(\mathbf{x}^k)(\mathbf{x}^k) + \mathbf{A}(\mathbf{x}^k)\,. \tag{16}$$

Details on the iterative methods for Richards' equation can be found, for example, in [5], [6].

4. Numerical Results

We present the numerical results for two types of stratified soil. In both cases the soil consists in two different porous media, sand and clay, that alternate in five equal width layers (0.20 m thickness). For each layer the hydraulic conductivity and water content function are given by the van Genuchten model [4]. The pressure head assigned at the boundaries, $h_{bot} = -2.0$m, $h_{up} = -0.5$m and at the initial time $h(0,z) + z = -2.0$m.

The figures represent the water content distribution (figs. 3a), 3b)) and the pressure head distribution (figs. 3c), 3d)) at two moments of time (4h and 48h).

Remark that at $t = 48$h, in the case with the sand on the top (fig. 3a) followed by the clay (the sand is more permeable than clay) the water content reaches the maximum along the first layer on the bottom, while in the case of the clay (fig. 3b) followed by the sand the maximum is attained on the top. Physicaly this means that the water is accumalating in the first case and it is draining in the second case.

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On recursively differentiable binary quasigroups

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Abstract. Complete 2-recursive codes which attains the Joshibound are constructed using recursively differentiable binary quasigroups. Some estimations of the maximum length of such codes are given.

Keywords: k-recursive code, strong orthogonal system of quasigroups, recursively differentiable quasigroups.

2000 Mathematics Subject Classification: 11T71

Consider a set $Q = \{a_1, a_2, ..., a_q\}$; $q, n \in N^*$ and let $C \subseteq Q^n$ be a code of length n over the alphabet Q. The code C is called a $[n, k, d]_Q$ - code if $|C| = q^k$ and the minimum Hamming distance of C is d. D.D.Joshi proved that the number $|C|$ of code words of a $[n, k, d]_Q$ - code does not exceed q^{n-d+1} [2]. Thus, say that a $[n, k, d]_Q$ - code C attains the Joshi bound if $|C| = q^{n-d+1}$. An open question is to describe the parameters q, n and d for which there exist $[n, k, d]_Q$ - codes with q^{n-d+1} code words, i.e. which attains the Joshi bound [1].It is known [2] that using strong orthogonal systems of finite k-ary quasigroups, in particular, orthogonal systems of latin squares, such codes can be constructed. For example, if $L_k = ||a_{ij}^k||$, $k = \overline{1,t}$, is an ortogonal system of latin squares, defined on a set $Q = \{1, 2, ..., q\}$, then

$$C = \{(i, j, a_{ij}^1, a_{ij}^2, ..., a_{ij}^t) \in Q^{t+2} |\ i, j = \overline{1,q}\}$$

is a $[t + 2, 2, t + 1]_Q$ - code which attains the Joshi bound [2].

Definition 1. *A code C of length n over an alphabet Q is called complete k-recursive, where $1 \leq k \leq n$, if there exists a mapping $f : Q^k \longrightarrow Q$, such that C consists of all code words $u = (u_0, u_1, ..., u_{n-1}) \in Q^n$ satisfying the conditions*

$$u_{i+k} = f(u_i, u_{i+1}, ..., u_{i+k-1}),\ i = \overline{0, n-k}.$$

A complete k-recursive code $C \subseteq Q^n$, defined by a mapping f, is denoted by $C(n, f)$.

Let C be a code of length n over an alphabet Q and $0 \leq i_0 < i_2 < ... < i_{k-1} \leq n - 1$. We say that C is a code with determining positions $i_0, i_1, ..., i_{k-1}$ if for every $c_0, c_1, ..., c_{k-1} \in Q$ there exists exactly one code word $u = (u_0, u_1, ..., u_{n-1}) \in C$ such that $u_{i_0} = c_0, u_{i_1} = c_1, ..., u_{i_{k-1}} = c_{k-1}$.

It is known [1] that a $[n, k, d]_Q$ - code C attains the Joshi bound iff every k positions in the code words of C are determining. A code $C \subseteq Q^n$ with determining positions $0, 1, ..., k-1$ can be described by a set of mappings $f_i : Q^k \longrightarrow Q,\ i = \overline{0, n-k-1}$, called the control functions, as follows:

$$(1) \qquad C = u \in Q_n |\ u = (u_0, u_1, ..., u_{k-1}, f_0(u_0^{k-1}), f_1(u_0^{k-1}), ..., f_{n-k-1}(u_0^{k-1}))$$

Definition 2. *A set of k-ary operations $\{A_1, A_2, ..., A_m\}_{m \geq 1}$, defined on a set Q, is called a strong orthogonal system if the system $\{E_1, E_2, ..., E_k, A_1, A_2,$
$..., A_m\}$, where $E_i(x_1^k) = x_i,\ i = \overline{1,k}$, is orthogonal.*

From the definition it follows that all operations in a strong orthogonal system $\{A_1, A_2, ..., A_m\}_{m\geq 1}$ are quasigroup operations. Moreover, every orthogonal system of binary quasigroups is a strong orthogonal system. It is proved in [1] that the code (1) attains the Joshi bound iff the system of k-ary operations $\{f_1, f_2, ..., f_{n-k-1}\}$ is strong orthogonal. If $C(n, f)$ is a complete k-recursive code then its positions $0, 1, ..., k-1$ are determining and the control functions $f^{(i)} : Q^k \longrightarrow Q,\ i = \overline{0, n-k-1}$, are the following

$$f^{(0)}(x_0^{k-1}) = f(x_0^{k-1}),$$
$$f^{(1)}(x_0^{k-1}) = f(x_1^{k-1}, f^{(0)}(x_0^{k-1})),$$
(2) .....
$$f^{(k-1)}(x_0^{k-1}) = f(x_{k-1}, f^{(0)}(x_0^{k-1}), f^{(1)}(x_0^{k-1}), ..., f^{(k-2)}(x_0^{k-1}))$$
$$f^{(i+k)}(x_0^{k-1}) = f(f^{(i)}(x_0^{k-1}), f^{(i+1)}(x_0^{k-1}), ..., f^{(i+k-1)}(x_0^{k-1})), \quad i \geq 0$$

The mapping $f^{(m)},\ m \in N$, defined in (2), is called the m-th k-recursive derivative of the mapping f. The 2-recursive derivatives $f^{(0)}, f^{(1)}, ..., f^{(t)}, t \in N$ of a binary quasigroup $Q(f)$ are quasigroup operations iff the complete 2-recursive code $C(t+3, f)$ attains the Joshi bound [1]. If $Q(f)$ is a k-ary quasigroup where $k \geq 3$, then the last result is not true.

Denote by $n(k, q)$ the maximum length of a $[n, k, d]_Q$-code, where $|Q| = q$ and $d = n - k + 1$ and by $n^r(k, q)$ the maximum length of a complete k-recursive code $C(n, f)$ which attains the Joshi bound. Then $n^r(k, q) \leq n(k, q)$ and for $q \geq 3, q$ - a primary number, we have $n^r(2, q) = n(2, q) = q + 1$ [1]. Evaluations of the parameters $n(k, q)$ and $n^r(k, q)$ are discussed in [1,2,4-6]. The condition $n^r(k, q) \geq t$ is equivalent to the existence of a k-ary quasigroup f on a set $Q, |Q| = q$, the first $t - k - 1$ k-recursive derivative of which form a strong orthogonal system.

Definition 3. *A binary quasigroup $Q(\cdot)$ is called recursively m-differentiable if its 2-recursive derivatives $f^{(1)}, f^{(2)}, ..., f^{(m)}$ are quasigroup operations.*

The problem of the complete evaluation of the parameters $n(k, q)$ and $n^r(k, q)$ is open. Denote by $N(k, q)$ the maximum number of k-ary quasigroups defined on a set Q, which form a strong orthogonal system. Then $n(k, q) = N(k, q) + k$. For $k = 2$ the number $n^r(2, q)$ is equal to the maximum natural integer t for which there exists a recursively $(t-3)$-differentiable quasigroup on a set Q with q elements.

If $Q(A)$ is a binary quasigroup, then its conjugates are defined as follows: $A^{-1}(x, y) = z \Leftrightarrow A(x, z) = y,\ {}^{-1}A(x, y) = z \Leftrightarrow A(z, y) = x,\ ({}^{-1}A)^{-1}(x, y) = z \Leftrightarrow A(y, z) = x,\ {}^{-1}(A^{-1})(x, y) = z \Leftrightarrow A(z, x) = y,\ A^*(x, y) = z \Leftrightarrow A(y, x) = z$ for every $x, y, z \in Q$. If the operation A is denoted by $(\cdot)$ then usually the conjugates ${}^{-1}A,\ A^{-1}$ and A^* are denoted by $(/),\ (\backslash)$ and $(*)$, respectively.

Proposition 4. *A binary quasigroup $Q(\cdot)$ is recursively 1-differentiable iff its conjugates $(*)$ and $(/)$ are orthogonal.*

Proof. Denote the first recursive derivative of $Q(\cdot)$ by $(\circ)$. Then $x \circ y = y \cdot xy$, for all $x, y \in Q$, and $Q(\circ)$ is a quasigroup iff each equation $x \circ a = b$ and $a \circ y = b$ has a unique solution for every $a, b \in Q$. Therefore, $Q(\cdot)$ is a quasigroup and $x \circ a = b \Leftrightarrow a \cdot xa = b$, then $Q(\circ)$ is a quasigroup iff the equation $a \circ y = b$ has a unique solution for every $a, b \in Q$. Thus as $a \circ y = b \Leftrightarrow y \cdot ay = b \Leftrightarrow y \setminus b = ay$, taking $y \setminus b = ay = c$ we get: $a \circ y = b \Leftrightarrow$

$$(y \setminus b = c \ \&\ \ a \cdot y = c) \Leftrightarrow (y \cdot c = b \ \&\ \ c/y = a) \Leftrightarrow (c * y = b \ \&\ \ c/y = a)$$

Hence the equation $a \circ y = b$ has a unique solution for every $a, b \in Q$ iff the conjugates " $*$ " and "$/$" of the quasigroup operation $(\cdot)$ are orthogonal. □

Corollary 5. *Every binary quasigroup $Q(\cdot)$ with the identity $xy \cdot x = y \cdot xy$ is a recursively 1-differentiable quasigroup.*

Note that every binary quasigroup $Q(\cdot)$ is orthogonal to its first recursive derivative $Q(\circ)$, but $Q(\circ)$ is not always a quasigroups. On the other hand the code

$$C(2, \cdot) = \{u \in Q^n |\ u = (x, y, x \cdot y, x \circ y)\}$$

attains the Joshi bound iff $Q(\circ)$ is a quasigroup.

Proposition 6. *A binary quasigroup $Q(A)$ is recursively 1-differentiable iff the equality*

$$^{-1}K(A(y,x)),^{-1}A(x,y)) = x,$$

holds for every $x, y \in Q$, where $K(x,y) = A((^{-1}A)^{-1}(x,y),x)$ is a quasigroup operation.

Proof. According to Mann's theorem [3] two binary quasigroups $Q(B)$ and $Q(C)$ are orthogonal iff there exists a quasigroup $Q(K)$ such that$^{-1}K(B(x,y),C(x,y)) = x$, for every $x, y \in Q$. Taking in the last equality $B(x,y) = A(y,x)$ and $C(x,y) =^{-1} A(x,y)$, we get $K(x,y) = A((^{-1}A)^{-1}(x,y),x)$, for every $x, y \in Q$. □

Corollary 7. *Every finite binary quasigroup $Q(\cdot)$ which satisfies the identity $xy \cdot x = y \cdot xy$ defines a complete 2-recursive idempotent code of length 4 which attains the Joshi bound.*

Corollary 8. *$n^{ir}(2,5^m) \geq 4$, for every $m \in N^*$.*

Proof. This estimation follows from Corollaries 1 and 2, using the existence of a quasigroup of order 5 (Table 1) with the identity $xy \cdot x = y \cdot xy$. □

	1	2	3	4	5
1	1	3	2	5	4
2	4	2	5	1	3
3	5	4	3	2	1
4	3	5	1	4	2
5	2	1	4	3	5

TABLE 1

Proposition 9. *A finite binary quasigroup $Q(\cdot)$ is recursively 2-differentiable iff*

(3) 1) $y \cdot xy = z \cdot xz \Leftrightarrow y = z$; 2) $xy \cdot (y \cdot xy) = xz \cdot (z \cdot xz) \Leftrightarrow y = z$

where $x, y, z \in Q$.

Proof. The quasigroup $Q(\cdot)$ is recursively 1-differentiable iff its first recursive derivative $Q(\circ)$ is a quasigroup. For a fixed element $a \in Q$ the following equivalences hold.

$$y \circ a = z \circ a \Leftrightarrow a \cdot ya = a \cdot za \Leftrightarrow y = z,$$

$$a \circ y = a \circ z \Leftrightarrow y \cdot ay = z \cdot az.$$

Thus, $Q(\circ)$ is a quasigroup iff (3) holds for every $a, y, z \in Q$.

The second recursive derivative $Q(\triangledown)$ of $Q(\cdot)$ is defined as follows: $x \triangledown y = xy \cdot (y \cdot xy)$, $x, y \in Q$. For every $a, b \in Q$, if the conditions 1) and 2) hold, we have $v \triangledown a = u \triangledown a \Leftrightarrow va \cdot (a \cdot va) = ua \cdot (a \cdot ua) \Leftrightarrow va = ua \Leftrightarrow v = u$. Moreover, $a \triangledown v = a \triangledown u \Leftrightarrow av \cdot (v \cdot av) = au \cdot (u \cdot au) \Leftrightarrow v = u$. Hence, as Q is a finite set we get that $Q(\triangledown)$ is a quasigroup, hence $Q(\cdot)$ is recursive 2-differentiable. □

Corollary 10. *Every commutative binary group of odd order is recursively 1-differentiable.*

Proof. Every group of odd order satisfies the condition $x^2 = y^2 \Rightarrow x = y$ and consequently satisfies (3). □

Proposition 11. *For every $q \equiv 0, 1$ or $3 (mod\ 4), q \geq 3$, there exists a complete 2-recursive code of length 4 over an alphabet Q, $|Q| = q$, which attains the Joshi bound.*

Proof. Let $F = GF(2^k)$, $k \in N \setminus \{0,1\}$ and $x \oplus y = \alpha x + \alpha y$, for all $x, y \in F$, where $\alpha \in F \setminus \{0, -1\}$. Then $F(\oplus)$ is a commutative quasigroup which satisfies the condition

$$x \oplus (x \oplus y) = z \oplus (z \oplus y) \Rightarrow y = z.$$

Hence $F(\circ)$, where $x \circ y = y \oplus (x \oplus y)$, for all $x, y \in F$, is a quasigroup orthogonal to $F(\oplus)$ and $|F| = q \equiv 0(mod\ 4)$. Moreover,

$$C(4, \oplus) = \{(x, y, x \oplus y, x \circ y) |\ x, y \in F\}$$

is complete 2-recursive and attains the Joshi bound.

Let now $H(\cdot)$ be an abelian group of odd order. Then $H(\cdot)$ satisfies the condition $x^2 = y^2 \Rightarrow x = y$ [2]. Thus $x \cdot (zx) = y \cdot (zy) \Rightarrow x^2 z = y^2 z \Rightarrow x^2 = y^2 \Rightarrow x = y$ in $H(\cdot)$, i.e. $H(\cdot)$ is orthogonal to its first recursive derivative $H(\circ)$. Hence the code

$$C(4, \cdot) = \{(x, y, x \cdot y, x \circ y) |\ x, y \in H\}$$

is a complete 2-recursive code which attains the Joshi bound and $|H| \equiv 1$ *or* $3(mod\ 4)$. □

Corollary 12. $n^r(2, q) \geq 4$ *for every odd* q, $q \geq 3$.

Proposition 13. *There exist recursively 2-differentiable binary quasigroups of order* q, *for every odd integer* q, $(q, 3) = 1$.

Proof. Let $Q(\cdot)$ be a group of odd order q, $(q, 3) = 1$. Then the mapping $x \to x^2$ is a bijection [2]. For $q \equiv 1(mod\ 3)$ there exists $k \in N : q = 3k + 1$, so $x^3 = y^3 \Rightarrow (x^3)^{k+1} = (y^3)^{k+1} \Rightarrow x^{3k+3} = y^{3k+3} \Rightarrow x^2 = y^2 \Rightarrow x = y$ i.e. the mapping $x \to x^3$ is a bijection. Similarly, for $q \equiv 2(mod\ 3)$, we get $x^3 = y^3 \Rightarrow x = y$, i.e. the mapping $x \to x^3$ is a bijection. If $Q(\cdot)$ is a commutative group and $Q(\circ)$ is its first recurrent derivative: $x \circ y = y \cdot xy$, $x, y \in Q$, then $x \circ y = x \cdot y^2$ therefore $Q(\circ)$ is an isotope of $Q(\cdot)$, i.e. is a quasigroup. If $Q(\triangledown)$ is the second recursive derivative of $Q(\cdot)$, then $x \triangledown y = x^2 \cdot y^3$ and $Q(\triangledown)$ is an isotope of $Q(\cdot)$ too, i.e. is a quasigroup. Hence the group $Q(\cdot)$ is recursively 2-differentiable. □

Corollary 14. $n^r(2, q) \geq 5$ *for every odd integer* q, $(q, 3) = 1$.

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Nonlinear operators with symmetry and potential branching equations

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Abstract. Under group symmetry conditions the variational type general problem of nonlinear equations branching theory without explicit separation of the linearization is considered. Generalizing some results in [1] about variational nonlinear equations with noninvariant kernel the cosymmetry identity is established, branching equation reduction possibilities are investigated and some consequences for symmetry breaking problems are obtained.
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1. Introduction

In the real Banach spaces E_1 and E_2 densely embedded into a Hilbert space H, $E_1 \subset E_2 \subset H$ the general problem of branching theory

$$F(x,\varepsilon) = 0, \quad F(x_0,0) = 0, \quad B_{x_0} = -F'_x(x_0,0) \tag{1}$$

is considered. Here the operator B_{x_0} is Fredholm continuously differentiable of x and sufficiently smooth of the small parameter ε in a neighbourhood of branching point $(x_0;0)$. The operator $F(\cdot,\varepsilon)$ allows the group G, i.e.

$$K_g F(x,\varepsilon) = F(L_g x,\varepsilon) \tag{2}$$

for its representations L_g in E_1 (K_g in E_2).

Throughout this paper it is supposed that the Lie group $G = G_r = G_r(a)$, $a = (a_1,\dots,a_r)$ is a r-dimensional differentiable manifold subject to the following requirements [1, 2, 3].

$\mathbf{c_1}$). the map $a \to L_{g(a)}x_0$, acting from a neighbourhood of the $G_r(a)$ identity element into E_1, belongs to the class C^1, such that $Xx_0 \in E_1$ for all infinitesimal operators $Xu = \lim\limits_{t\to 0} t^{-1}\left[L_{g(a(t))}u - u\right]$ in the tangent manifold $T^r_{g(a)}$ to $L_{g(a)}$;

$\mathbf{c_2}$). the stationary subgroup of x_0 determines the representation $L(G_s)$ of the local Lie group $G_s \subset G_r$, $s < r$ with the s-dimensional subalgebra $T^s_{g(a)}$ of infinitesimal operators. Thus, the elements $\varphi = Xx_0$, $X \in T^r_{g(a)}$, determine in $\mathcal{N}(B_{x_0})$ an $m = r - s$-dimensional subspace, i. e. the bases in $\mathcal{N}(B_{x_0})$ and in the Lie subalgebra $T^r_{g(a)}$ can be ordered such that $\varphi_k = \varphi_k(x_0) = X_k x_0$, $1 \le k \le m$ and $X_k x_0 = 0$ for $k \ge m+1$.;

$\mathbf{c_3}$). for every $X \in T^r_{g(a)}$ the map $X : E_1 \to H$ is bounded in the $\mathcal{L}(E_1,H)$–topology.

Definition 1. *$F(x,\varepsilon)$ is an operator of potential type if in a certain neighbourhood of the branching point $(x_0;0)$*

$$F(x,\varepsilon) = D\,grad\, l\,(x,\varepsilon) \tag{3}$$

where $D \in \mathcal{L}(E_2)$ is an invertible operator. Then the functional $l\,(x,\varepsilon)$ is called the pseudopotential for the potential type operator F.

The aim of this article is to generalize the results in [1, 2] to the potential type operators.

Differentiating equality (2) with respect to x at the point x_0 we obtain $K_g F'_x(x_0,\varepsilon) = F'_x(L_g x_0,\varepsilon)L_g$, i.e. the kernel of the Fredholm operator B_{x_0} turns out to be only G_s-invariant. The analog of the cosymmetric identity given in [1, 2] is obtained and on its base a theorem concerning possibility to reduce the branching equation is proved. As consequences, for the case of the invariant kernel $\mathcal{N}(B_{x_0})$, results on the branching equation (BEq) for some symmetry breaking problems are established.

2. Cosymmetric Identity and BEq Reduction

For our problem the BEq has the form

$$f(\xi,\varepsilon)\equiv QF(x_0+v(\varepsilon)+u(v(\varepsilon),\varepsilon),\varepsilon),\ v(\varepsilon)=v(x_0,\varepsilon)=P(x-x_0)=\sum_{i=1}^{n}\xi_i(\varepsilon)\varphi_i, \tag{4}$$

where Q is the projector on the direct supplement $E_{2,n}=\operatorname{span}\{z_1,\dots,z_n\}$ of the range $\mathcal{R}(B_{x_0})$ to the operator B_{x_0} [4], P is the projector on $\mathcal{N}(B_{x_0})=\operatorname{span}\{\varphi_1,\dots,\varphi_n\}$.

Lemma 2. *Let the conditions* $\mathbf{c_1}$)–$\mathbf{c_3}$) *be satisfied and assume that the equation (1) is of potential type (3). Its pseudopotential is invariant relative to representation* $L_{g(a)}$ *iff*

$$L_g^*D^{-1}=D^{-1}K_g^{-1}\quad or\quad DL_g^{*-1}=K_gD. \tag{5}$$

Proof. According to (3) and by Lagrange's theorem

$$l(x,\varepsilon)=\int_0^1\left\langle D^{-1}F(\theta x,\varepsilon),x\right\rangle_H d\theta,\qquad l(L_gx,\varepsilon)=\int_0^1\left\langle D^{-1}F(L_g\theta x,\varepsilon),L_gx\right\rangle_H d\theta.$$

Since $Dgrad\,l(x,\varepsilon)=K_g^{-1}F(L_g\theta x,\varepsilon)$, then by Lagrange theorem, we have

$$l(x,\varepsilon)=\int_0^1\left\langle D^{-1}K_g^{-1}F(L_g\theta x,\varepsilon),x\right\rangle_H d\theta.$$

Consequently the invariance of the pseudopotential $l(x,\varepsilon)$ is possible iff

$$\int_0^1\left\langle (L_g^*D^{-1}-D^{-1}K_g^{-1})F(L_g\theta x,\varepsilon),x\right\rangle_H d\theta=0,$$

□

Corollary 3. *For the potential nonlinear equation (1) we have* $L_g^*K_g=I$.

Theorem 4. *For all* $X\in T^r_{g(a)}$ *in some neighbourhood of the branching point* $(x_0;0)$ *the cosymmetric identity holds*

$$\langle DF(x,\varepsilon),X(x,\varepsilon))\rangle_H=0,\quad x=x_0+v(\varepsilon)+u(v(\varepsilon),\varepsilon). \tag{6}$$

Proof. Let be $X\in T^r_{g(a)}$ and let $L_{g(a(t))}$ be one-parameter subgroup of $L_{g(a)}$. According to the functional $l(x,\varepsilon)$ L_g–invariance one has

$$0=l\left(L_{g(a(t))}x,\varepsilon\right)-l\left(x,\varepsilon\right)=\left\langle D^{-1}F(x,\varepsilon),(L_{g(a(t))}-I)x\right\rangle_H+o\left(\|(L_{g(a(t))}-I)x\|_{E_1}\right),$$

where $x=x_0+v(\varepsilon)+u(v(\varepsilon),\varepsilon)$. Passage to limit $t\to 0$ proves the theorem. □

Theorem 5. *Under the conditions* $\mathbf{c_1}$) – $\mathbf{c_3}$) *assume that the equation (1) is of potential type, its* L_g*–invariant pseudopotential belongs to the* C^2 *class in a neighbourhood of the branching point* $(x_0;0)$*, and let* s *be the dimension of the stationary subgroup of the element* x_0 *with* $m=r-s>0$.

1. *If* $m=n$*, then for all* $(\xi(\varepsilon),\varepsilon)$ *(or* $(v(x_0,\xi(\varepsilon)),\varepsilon)$*) belonging to some small neighbourhood of* $0\in\mathbb{R}^{n+1}$ *the BEq (4) is identically satisfied.*
2. *If* $m<n$ *and* $n\geq 2$*, then the partial reduction of the BEq takes place:* m *of its equations are linear combinations of the other* $(n-m)$ *ones.*

Proof. According to the cosymmetric identity and by Lyapounov–Schmidt method [4, 5] we have

$$\begin{aligned}0=&\langle DF(x_0+v(\varepsilon)+u(v(\varepsilon),\varepsilon),\varepsilon),X_k(x_0+v(\varepsilon)+u(v(\varepsilon),\varepsilon))\rangle_H=\\&\langle DQF(x_0+v(\varepsilon)+u(v(\varepsilon),\varepsilon),\varepsilon),X_k(x_0+v(\varepsilon)+u(v(\varepsilon),\varepsilon))\rangle_H=\\&=\sum_{j=1}^{n}f_j(\xi,\varepsilon)\left[\langle Dz_j,\varphi_k\rangle_H+\langle Dz_j,X_k(v(\varepsilon)+u(v(\varepsilon),\varepsilon))\rangle_H\right],\end{aligned}$$

where the rank of the $(n\times m)$–matrix $\left[\langle Dz_j,\varphi_k\rangle_H\right]_{j=\overline{1,n},\ k=\overline{1,m}}$ is equal to m. The theorem is proved. $\square$

3. Application to symmetry breaking problems

If the kernel $\mathcal{N}(B_{x_0})$ is L_g-invariant, i.e.$L_{g(a)}x_0 = x_0$, the cosymmetric identity (6) gives a new approach to the problem of constructing the general form of the BEq which allows a group symmetry [6]. In the following some examples of potential and partially potential branching equations with symmetries of rotation and hyperbolic rotation groups are given.

Theorem 6. [6] *The two–dimensional analytic potential BEq with symmetry $SO(2)$ $(SH(2))$ allows the symmetry $O(2)$ $(H(2))$. Moreover, in the real base in $\mathcal{N}(B_{x_0})$ we have $v(\tau,\varepsilon)=\sum\limits_{k=1}^{2}\tau_k(\varepsilon)\varphi_k$ and the BEq has the form*

$$
\begin{gathered}
f_1(\tau,\varepsilon)\equiv\sum_{k=0}^{\infty}a_k(\varepsilon)(\tau_1^2\pm\tau_2^2)^k\tau_1=0,\quad f_2(\tau,\varepsilon)\equiv\sum_{k=0}^{\infty}a_k(\varepsilon)(\tau_1^2\pm\tau_2^2)^k\tau_2=0,\\
l(\tau,\varepsilon)=\sum_{k=0}^{\infty}\tfrac{a_k(\varepsilon)}{2(k+1)}\left(\tau_1^2\pm\tau_2^2\right)^{k+1},
\end{gathered}
\tag{7}
$$

("+" stands for the $O(2)$– and "-" for the $H(2)$–symmetry).

Theorem 7. *The $2p$-dimensional analytic BEq with symmetry of p-dimensional cube and symmetries $SO(2)$ $(SH(2))$ in the i-th pair of variables τ_{2i-1},τ_{2i} at independent group parameters for different i-s has the form*

$$
\begin{aligned}
f_{2k-1}(\tau,\varepsilon) &\equiv \tau_{2k-1}u\left(\underset{k}{|\tau|},\underset{2}{|\tau|},\dots,\underset{k-1}{|\tau|},\underset{1}{|\tau|},\underset{k+1}{|\tau|},\dots,\underset{l}{|\tau|},\varepsilon\right)=0,\\
f_{2k}(\tau,\varepsilon) &\equiv \tau_{2k}u\left(\underset{k}{|\tau|},\underset{2}{|\tau|},\dots,\underset{k-1}{|\tau|},\underset{1}{|\tau|},\underset{k+1}{|\tau|},\dots,\underset{l}{|\tau|},\varepsilon\right)=0,
\end{aligned}
\tag{8}
$$

where $\underset{k}{|\tau|}=\tau_{2k-1}^2\pm\tau_{2k}^2$, $k=1,\dots,l$. The function u is invariant relative to pairwise transpositions of arguments with indexes $k>1$. If $\underset{1}{l}=l\left(\underset{1}{|\tau|},\underset{2}{|\tau|},\dots,\underset{l}{|\tau|},\varepsilon\right)$ is the partial potential (pseudopotential) of the first pair of equations on the variables τ_1,τ_2, then $\underset{k}{l}=l\left(\underset{k}{|\tau|},\dots,\underset{k-1}{|\tau|},\underset{1}{|\tau|},\underset{k+1}{|\tau|},\dots,\underset{l}{|\tau|},\varepsilon\right)$ is the potential (pseudopotential) of the k-th pair and

$$
l\left(\underset{1}{|\tau|},\dots,\underset{i}{|\tau|},\dots,\underset{j}{|\tau|},\dots,\underset{l}{|\tau|},\varepsilon\right)=l\left(\underset{1}{|\tau|},\dots,\underset{j}{|\tau|},\dots,\underset{i}{|\tau|},\dots,\underset{l}{|\tau|},\varepsilon\right).
$$

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Discrete groups of W-symmetry

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Abstract. One of the essential generalizations of the classical symmetry is P-symmetry of A.M.Zamorzaev. In the case of P-symmetry the transformations of the qualities, attributed to the points of geometrical figure F with discrete symmetry group G, are combined directly with the geometrical transformations and do not depend on the choice of points. In the case of W-symmetry the transformations of the qualities, attributed to the points, essentially depend on the choice of points. The discrete groups of W_p-symmetry are subgroups of left standard Cartesian wreath product of initial group P with group G, while the discrete groups of W_q-symmetry are subgroups of the crossed standard Cartesian wreath product of group P with G. The methods of deriving the groups of W_p-symmetry and W_q-symmetry demands the generalization of homomorphisms as the natural left quasihomomorphism and, respectively, as the crossed quasihomomorphism. Moreover, it requires the investigation of some their properties.

1. Introduction

The theory of symmetry of the real crystal gives rise to new generalizations of classical symmetry: antisymmetry, multiple antisymmetry, coloured symmetry, magnetic symmetry, cryptosymmetry etc. One of the essential generalizations of the Shubnikov's antisymmetry [1] and of the Belov's colour symmetry [2] is the P-symetry of Zamorzaev [3-5]. We discuss briefly the essence of the P-symmetry and the $\overline{P}$-symmetry.

Associate with each point of the geometrical figure F with discrete symmetry group G at least one index from the set $N = \{1, 2, ..., m\}$ and fix a certain transitive group P of the permutations of these indexes. The transformation of P-symmetry is defined to be an isometric mapping $g^{(p)} = gp = pg$ of the "indexed" geometrical figure $F^{(N)}$ onto itself in which the geometrical component g operates only on points, and the indexes are transformed by the permutation p of the group P. The set $G^{(P)}$ of transformations of P-symmetry of any "indexed" geometrical figure $F^{(N)}$ forms a group with respect to the operation

$$g_i^{(p_i)} \star g_j^{(p_j)} = g_k^{(p_k)}, \tag{1}$$

where $g_k = g_i g_j$ and $p_k = p_i p_j$ [3].

The set $P' = \{p | g^{(p)} \in G^{(P)}\}$ is a subgroup of the group P. When $P' = P$, the group $G^{(P)}$ is defined as the group of complete P-symmetry. Identifying the groups G and P with their isomorfic injections into $G \times P$ by the rules $g \mapsto ge$, where e is the unit of the group P, and $p \mapsto 1p$, where 1 is the unit of the group G, we find the symmetry subgroup $H = G^{(P)} \cap G$ and the subgroup $Q = G^{(P)} \cap P = G^{(P)} \cap P'$ of P-identical transformations of the group $G^{(P)}$. The group $G^{(P)}$ is called major, minor or Q-middle if $e < Q = P' = P, e = Q < P' = P$ or $e < Q < P' = P$, respectively. If P' is a non-trivial subgroup of P, then the group $G^{(P)}$ is called P'-semi-major, P'-semi-minor or (P', Q)-semi-middle according to the cases when $e < Q = P', e = Q < P'$ or $e < Q < P'$.

Every such group of complete P-symmetry can be derived from the generating group G by searching in G and P for invariant subgroups H and Q for which there is an isomorphism of factor-groups G/H and P/Q, by paired multiplication of the cosets adequate in isomorphism and by joining the obtained products [3-5]. The cases when $Q = P$, $Q = e$ and $e < Q < P$ make possible to divide the groups into major, minor and Q-middle ones.

In the scheme of the P-symmetry the Belov's colour symmetry (m-symmetry) corresponds to the cyclic group $P = \{(12...m)\}$, the Shubnikov's antisymmetry is the 2-symmetry, the l-multiple antisymmetry is the $(2, 2, ..., 2)$-symmetry, the Neronova-Belov's colour antisymmetry is the $(m, 2)$-symmetry

etc. The P groups can be interpreted as vertices permutations at the symmetry of an oriented regular m-gon in the case of the m-symmetry.

On the other hand, the Q-symmetry techniques suggested by Koptsik [6] and his collaborators under the influence of the Tavger-Zaitsev magnetic symmetry description [7] (in which also unlike the P-symmetry techniques the geometric transformation component acts both on the points and on the features bearing not scalar but vector or tensor nature) was developed in detail by the author of this article and called a $\overline{P}$-symmetry [8,9,5].

The transformation of $\overline{P}$-symmetry is defined as an isometric mapping $g^{(p)} = pg$ of the "indexed" geometrical figure $F^{(N)}$ onto itself such that the geometrical component g operates both on points and on indexes by the given rule which is independent of the points however the permutation p is only a compensating permutation of indexes to map $F^{(N)}$ onto itself and $p \in P$. In this case, in general, the components p and g of the transformation $g^{(p)}$ do not commute: $pg \neq gp$.

The set $G^{(\overline{P})}$ of transformations of $\overline{P}$-symmetry of any "indexed" geometrical figure $F^{(N)}$ forms a group with respect to the operation

$$(2) \qquad g_i^{(p_i)} * g_j^{(p_j)} = g_k^{(p_k)},$$

where $g_k = g_i g_j$, $p_k = p_i \overrightarrow{\varphi_{g_i}}(p_j)$ and $\overrightarrow{\varphi_{g_i}}(p_j) = g_i p_j g_i^{-1} = p_s \in P$ [8]. The groups of $\overline{P}$-symmetry are subgroups of the right semi-direct products of the group P by group G, accompanied with homomorphism $\varphi : G \to AutP$ by the rule $\varphi(g_i) = \overrightarrow{\varphi_{g_i}}$. The set $P' = \{p | g^{(p)} \in G^{(\overline{P})}\}$ is a subset of the group P. Moreover, $e \subseteq P' \subseteq P$. If $P' \subset P$, but P' is not a group, then the group $G^{(\overline{P})}$ is called a P'-pseudo-minor or (P', Q)-pseudo-middle, if $e = Q \subset P'$ or $e < Q \subset P'$.

2. Discrete groups of W_p-symmetry

Construct the Cartesian product W of isomorphic copies of the group P which are indexed by elements of G, i.e. $W = \overline{\prod}_{g_i \in G} P^{g_i}$, where $P^{g_i} \cong P$. The transformation of W_p-symmetry is by definition an isometric mapping $g^{(w)} = gw$ of the "indexed" geometrical figure $F^{(N)}$ onto itself in which the geometrical component g operates only on points $M_k = g_k(M_1)$ of the figure $F^{(N)}$ (where M_1 is a fixed point of general position of the figure F with respect to the group G), not affecting indexes, and the indexes ascribed to the point M_k are transformed by the permutation p^{g_k}, which is the "g_k-component" in w. The set of transformations of W_p-symmetry of the given "indexed" geometrical figure $F^{(N)}$ forms a group $G^{(W_p)}$ with the operation

$$(3) \qquad g_i^{(w_i)} \circ g_j^{(w_j)} = g_k^{(w_k)},$$

where $g_k = g_i g_j$, $w_k = w_i^{g_j} w_j$ and $w_i^{g_j}(g_s) = w_i(g_j g_s)$.

The classification of groups of W_p-symmetry on types [10] is similar to the classification of groups of $\overline{P}$-symmetry. The groups of W_p-symmetry are subgroups of the left simi-direct products of group W with the group G, accompanied with a fixed isomorphism $\varphi : G \to AutW$ defined by the rule $\varphi(g) = \overleftarrow{g}$, where $\overleftarrow{g} : w \mapsto w^g$.

Let us have groups G, P and $W = \overline{\prod}_{g_i \in G} P^{g_i}$, where $P^{g_i} \cong P$, the exact isomorphic injection $\varphi : G \to AutW$ by the rule $\varphi(g) = \overleftarrow{g}$, where $\overleftarrow{g} : w \mapsto w^g$. The mapping μ of the group G onto the subset W' of the group W defined by the rule $\mu(g) = w$ is called an *exact natural left quasi-homomorphism* if for any g_i and g_j from G we have

$$\mu(g_i g_j) = [\mu(g_i)]^{g_j} \mu(g_j) = w_i^{g_j} w_j = w_k,$$

where $w_i, w_j, w_k \in W'$ [11].

In general, the image of G through the exact natural left quasi-homomorphism μ, $\mu(G) = W\prime \subset W$ is not a group, but $W\prime$ always contains the unit w_0 of the group W. The kernel H of the exact natural left quasi-homomorphism μ of the group G into the group W is a subgroup in G; the index of this subgroup coincides with the power of $\mu(G)$.

Let us consider the group G, the finite group $W = \prod_{g_i \in G} P^{g_i}$, its subgroup V ($V < W$) and the exact isomorphic injection φ of the group G into the subgroup $\overleftarrow{G}$ of the group $AutW$ defined by the rule $\varphi(g) = \overleftarrow{g}$. The mapping $\tilde{\mu}$ of the group G onto the subset X of the set of all left cosets of group W by its subgroup V is called a *generalized exact natural left quasi-homomorphism* if for any g_i and g_j from G the conditions $\tilde{\mu}(g_i) = w_i V$ and $\tilde{\mu}(g_j) = w_j V$ imply

$$\tilde{\mu}(g_i g_j) = (w_i V)^{g_j} * w_j V = w_k V,$$

where $w_i V, w_j V, w_k V \in X$.

The mapping $\tilde{\mu}$ of the group G onto the subset X of the set of all left cosets of the finite group $W = \prod_{g_i \in G} P^{g_i}$ by its subgroup V defined by the rule $\tilde{\mu}(g) = wV$ is a generalized exact natural left quasi-homomorphism iff $V^g = wVw^{-1}$ for any $g \in G$ and $wV = \tilde{\mu}(g)$ [12].

Any group $G^{(W_p)}$ of W_p-symmetry with the finite generating group G and group $W = \prod_{g_i \in G} P^{g_i}$ of multicomponent permutations by the following steps:

1) find in W all subgroups V and subsets W', which are decomposed in left cosets by its subgroup V, and in G all proper subgroups H with the index equal to the power of set of all left cosets of W' by V and for which there is the isomorphism λ of factor-groups G_1/H and W_1/V_1 ($\lambda : G_1/H \to W_1/V_1$ defined by the rule $\lambda(Hg) = wV$), where $G_1 \leq G$, $W_1 \leq DiagW$ and $V_1 = V \cap DiagW \leq W_1$;
2) construct a generalized exact natural left quasi-homomorphism $\tilde{\mu}$ of the group G onto the set of all left cosets of W' by the subgroup V defined by the rule $\tilde{\mu}(Hg) = wV$ and which preserves the correspondence between the elements of factor-groups G_1/H and W_1/V_1 obtained as a result of isomorphism λ;
3) combine pairwise each g' of Hg with each w' of $wV = \tilde{\mu}(g')$;
4) introduce into the set of all these pairs the operation (3) [13].

If $V = w_0$, where w_0 is the unit of the group W, then the mapping $\tilde{\mu}$ is an ordinary exact natural left quasi-homomorphism. In this case, the universal method of deriving the groups of W_p-symmetry becomes more simple and takes the form of method for deriving the semi-minor or pseudo-minor groups in dependence on W', where $w_0 \subset W' \subset W$.

3. Discrete groups of W_q-symmetry

The transformation of W_q-symmetry is defined to be an isometric mapping $g^{(w)} = wg$ of the "indexed" geometrical figure $F^{(N)}$ onto itself in which the geometrical component g operates both on points $M_k = g_k(M_1)$ of the figure $F^{(N)}$ (where M_1 is a point of general position of the figure F with respect to the group G) and on indexes by the given rule independent of the points, but the permutation p^{g_k} ("g_k-component" in w) is only a compensating permutation of indexes at the point M_k to map $F^{(N)}$ onto itself. The set $G^{(W_q)}$ of transformations of W_q-symmetry of the given "indexed" figure $F^{(N)}$ forms a group with the operation

$$g_i^{(w_i)} \otimes g_j^{(w_j)} = g_k^{(w_k)}, \tag{4}$$

where $g_k = g_i g_j$, $w_k = w_i^{g_j} \overrightarrow{\tau_{g_i}}(w_j), w_i^{g_j}(g_s) = w_i(g_j g_s)$ and $\overrightarrow{\tau_{g_i}}(w_j) = g_i w_j g_i^{-1}$.

The groups of W_q-symmetry are subgroups of the crossed semi-direct products of the group $W = \overline{\prod}_{g_i \in G} P^{g_i}$ with the generating group G, accompanied with two morphisms: a homomorphism $\tau : G \to AutW$ defined by the rule $\tau(g) = \overrightarrow{\tau_g}$, where $\overrightarrow{\tau_g}(w) = gwg^{-1}$, and an isomorphism $\varphi : G \to AutW$ defined by the rule $\varphi(g) = \overleftarrow{g}$, where $\overleftarrow{g} : w \mapsto w^g$ and $w^g(g_s) = w(gg_s)$ [14].

Let us consider groups G, P and $W = \overline{\prod}_{g_i \in G} P^{g_i}$ (where $P^{g_i} \cong P$), the isomorphic injection $\varphi : G \to AutW$ defined by the rule $\varphi(g) = \overleftarrow{g}$, where the automorphism $\overleftarrow{g}$ makes the left g-translation of the components in $w \in W$ (i.e. $\overleftarrow{g} : w \mapsto w^g$), and also the homomorphism $\tau : G \to \Phi \leq AutW$ (where $\tau(g) = \overrightarrow{\tau_g}$ and $\overrightarrow{\tau_g}(w) = gwg^{-1}$). The mapping α of the group G onto the subset W' of the group

W defined by the rule $\alpha(g) = w$ is called a *crossed quasi-homomorphism*, accompanied by exact left translation of components and by homomorphism τ of right conjugation, if for any g_i and g_j from G

$$\alpha(g_i g_j) = [\alpha(g_i)]^{g_j} * \overrightarrow{\tau}_{g_i} [\alpha(g_j)] = w_i^{g_j} \overrightarrow{\tau}_{g_i} (w_j) = w_k,$$

where $w_i, w_j, w_k \in W'$.

We note that in the case of $\overrightarrow{\tau}_g = i$ (where i is the identical automorphism of the group W for any g from G) the crossed quasi-homomorphism α is an ordinary exact natural left quasi-homomorphism; if $w^g = w$ for all $g \in G$ and $w \in \alpha(G)$, then α is right quasi-homomorphism accompanied by homomorphism τ of right conjugation.

Any semi-minor group $G^{(W_q)}$ of W_q-symmetry can be derived from its generating group G and group $W = \overline{\prod}_{g_i \in G} P^{g_i}$, knowing the kernel H_1 of accompanying homomorphism $\tau : G \to AutW$ of right conjugation, by the following steps: 1) to construct a crossed quasi-homomorphism α of the group G onto the non-trivial subgroup W' of W $(w_0 < W' < W)$ by the rule $\alpha(g) = w$; 2) to combine pairwise each g of G with each $w = \alpha(g)$; 3) to introduce into the set of all these pairs the operation (4).

Let us consider the group G, the finite group $W = \prod_{g_i \in G} P^{g_i}$ (where $P^{g_i} \cong P$), its subgroup V, the isomorphic injection $\varphi : G \to AutW$ defined by the rule $\varphi(g) = \overleftarrow{g}$ (where $\overleftarrow{g}$: $w \mapsto w^g$) and also the homomorphism $\tau : G \to AutW$ (where $\tau(g) = \overrightarrow{\tau}_g$ and $\overrightarrow{\tau}_g (w) = gwg^{-1}$). The mapping $\tilde{\alpha}$ of the group G onto the subset X of the set of all right cosets of group W by its subgroup V is caled a *generalized crossed quasi-homomorphism*, accompanied by the exact left translation of components and by the homomorphism τ of right conjugation, if for any g_i and g_j from G from the conditions $\tilde{\alpha}(g_i) = Vw_i$ and $\tilde{\alpha}(g_j) = Vw_j$ it follows that

$$\tilde{\alpha}(g_i g_j) = (Vw_i)^{g_j} * \overrightarrow{\tau}_{g_i} (Vw_j) = Vw_k,$$

where $Vw_i,\ Vw_j,\ Vw_k \in X$ and $\overrightarrow{\tau}_{g_i} = \tau(g_i) \in \Phi \leq AutW$.

If $V = w_o$, the generalized crossed quasi-homomorphism $\tilde{\alpha}$ is an ordinary crossed quasi-homomorphism. If $\overrightarrow{\tau}_g = i$ for any g from G, then the mapping $\tilde{\alpha}$ is a generalized exact natural left quasi-homomorphism. Moreover, if $w^g = w$ for any g from G and w from $W' = \tilde{\alpha}(G)$, then $\tilde{\alpha}$ is a generalized right quasi-homomorphism accompanied by the homomorphism τ of right conjugation.

Any middle group $G^{(W_q)}$ of W_q-symmetry with the finite group W and the subgroup V of W-identical transformations can be derived from its finite generating group G and group $W = \prod_{g_i \in G} P^{g_i}$ of multicomponent permutations by the following steps:

1) find in W all proper $\overleftarrow{G}$-invariant subgroups V [15] $(w_0 < V < W)$;
2) construct a generalized crossed quasi-homomorphism $\tilde{\alpha}$ of the group G onto the set of all right cosets of the group W by the subgroup V defined by the rule $\tilde{\alpha}(g) = Vw$;
3) combine pairwise each g of G with each w' of $Vw = \tilde{\alpha}(g)$;
4) introduce into the set of all these pairs the operation (4).

The theory of P-, $\overline{P}$-, W_p- and W_q-symmetry groups was elaborated and developed by Zamorzaev's geometrical school from Chisinau.

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Lagrangian geometrization of electrodynamics

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Abstract. One studies the Lagrange space of electrodynamics $L^n = (M, L)$, L being defined in (1). The Euler–Lagrange equations and the law of conservation of energy is pointed out. The Lorentz equations (8) are used in order to determine the canonical nonlinear connection and the canonical metrical connection of space L^n, as well as to study the Maxwell and Einstein equations of L^n.

Introduction

The notion of Lagrange space, whose geometry, introduced and studied by the author, [1, 2, 3], is defined as a pair $L^n = (M, L(x, y))$, where M is the configurations space and $L : TM \to R$ is a regular Lagrangian, TM being the phase space.

In the present paper we apply the geometry of the Lagrange space L^n to the case of the known Lagrangian of electrodynamics

$$L(x, \dot{x}) = mc\gamma_{ij}(x)\dot{x}^i\dot{x}^j + \frac{2e}{m}A_i(x)\dot{x}^i + U(x),$$

where m, c, e are known physical constants, $\gamma_{ij}(x)$ is a pseudo–Riemannian metric (the gravitational potentials), $A_i(x)$ is the electromagnetic covariant vector field (the electromagnetic potentials) and $U(x)$ is a potential function.

The space $L^n = (M, L(x, \dot{x}))$ is called the Lagrange space of electrodynamics. An introduction in the geometry of these spaces may be found in the books [2, 3].

Now we emphasize the general theory of the space L^n. The main geometrical object fields of the Lagrange space of electrodynamics: canonical nonlinear connection N, canonical N-metrical connection, the Maxwell and Lagrange equations for electromagnetic and gravitational fields are pointed out.

1. The Lagrange space of electrodynamics

Consider the Lagrange space $L^n = (M, L(x, y))$ with the fundamental function $L(x, y)$ of electrodynamics

$$L(x, y) = mc\gamma_{ij}(x)y^iy^j + \frac{2e}{m}A_i(x)y^i + U(x). \tag{1}$$

The local coordinates of a point $(x, y) \in TM$ are (x^i, y^i). The indices $i, j, ...$ run over the set $\{1, 2, ..., n\}$ and the Einstein's convention of summation is used. The number n is the dimension of differentiable manifold M.

The metric tensor $g_{ij}(x, y)$ of the space L^n is $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$. It is given by

$$g_{ij}(x, y) = mc\gamma_{ij}(x). \tag{2}$$

Of course, its contravariant g^{ij} is expressed by

$$g^{ij}(x, y) = \frac{1}{mc}\gamma^{ij}(x). \tag{3}$$

The first important result is obtained using the variational calculus for the action integral of the Lagrangian $L(x, y)$

$$I(c) = \int_0^1 L(x, \dot{x})dt.$$

We obtain the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad y^i = \frac{dx^i}{dt}. \tag{4}$$

In order to write these equations for the Lagrangian (1), we adopt the following notations: $\gamma^i_{jk}(x)$ are the Christoffel symbols for the fundamental tensor g_{ij}. They are same for $\gamma_{ij}(x)$. Also, we denote:

$$F_{ij}(x) = \tfrac{e}{2m}\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right) \tag{5}$$
$$F^i{}_j(x) = g^{ik}(x)F_{kj}(x),$$

where $F_{ij}(x)$ is the electromagnetic tensor field of the space L^n.

We can prove, without difficulty

Theorem 1. *The Euler–Lagrange equations (4) have the following expression:*

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x)\,\frac{dx^j}{dt}\,\frac{dx^k}{dt} = g^{ij}(x)\left[eft[F_{jk}(x)\,\frac{dx^k}{dt} + \frac{1}{2}\frac{\partial U}{\partial x^j}\right]. \tag{6}$$

The solution curves of the differential equations (6) are called the extremal curves of the space L^n.

The *energy* of the Lagrangian $L(x,y)$, from (1) is $\mathcal{E}_L = y^i\,\dfrac{\partial L}{\partial y^i} - L$. We have

$$\mathcal{E}_L(x,y) = g_{ij}(x)y^iy^j - U(x). \tag{7}$$

The scalar function $g_{ij}(x)y^iy^j$ is the so–called *absolute energy* of the space L^n. It is related to the kinetic energy $T(x,\dot{x}) = \dfrac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$.

Taking into account a general result [2, 4], we have the known law of conservation:

Theorem 2. *The energy $\mathcal{E}_L$ of the Lagrange space of electrodynamics L^n is constant along every extremal curve of L^n.*

Two particular cases are remarkable:

1° if the potential function $U(x)$ is constant, then the Euler–Lagrange equations (6) are the Lorentz equations of the space L^n:

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x)\,\frac{dx^j}{dt}\,\frac{dx^k}{dt} = F^i{}_j(x)\,\frac{dx^j}{dt}; \tag{8}$$

2° if the function $U(x)$ is constant and the covector $A_i(x)$ is the gradient of a function $\varphi(x)$, then the Euler–Lagrange equations of the space L^n are

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x)\,\frac{dx^j}{dt}\,\frac{dx^k}{dt} = 0 \tag{9}$$

and the extremal curves are the geodesics of the pseudo–Riemannian space $(M, \gamma_{ij}(x))$.

2. Canonical non–linear connection

The canonical non–linear connection N of the Lagrange space of electrodynamics can be obtained from the Lorentz equations (8). Indeed, taking into account the fact that (8) determines a semispray S

$$S = y^i\,\frac{\partial}{\partial x^i} - 2G^i(x,y)\,\frac{\partial}{\partial y^i}$$

with the coefficients

$$2G^i = \gamma^i_{jk}(x)y^jy^k - F^i{}_k(x)y^k$$

it follows [2] that the coefficients $N^i{}_j$ of N are

$$N^i{}_j = \gamma^i_{jk}(x)y^k - F^i{}_k(x). \tag{10}$$

But, N can be considered as a supplementary distribution to the vertical distribution V, i.e. we have

$$T_u(TM) = N(u) \oplus V(u), \quad \forall\, u \in TM. \tag{11}$$

An adapted basis to the distributions N and V is determined, locally, by $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, $(i = 1, ..., n)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i(x, y)\ \frac{\partial}{\partial y^j} \tag{12}$$

The dual basis of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ is $(dx^i, \delta y^i)$, $(i = 1, ..., n)$ with

$$\delta y^i = dy^i + N^i{}_j(x, y)dx^j. \tag{13}$$

Now, we can investigate the case when the canonical nonlinear connection N is integrable.

Theorem 3. *The canonical nonlinear connection N of the space L^n is integrable iff the pseudo–Riemannian space $\mathcal{R}^n = (M, \gamma_{ij}(x))$ is flat and the electromagnetic tensor $F^i{}_j(x)$ is absolutely parallel with respect to the Levi–Civita connection $\gamma^i_{jk}(x)$.*

Proof. The tensor of integrability $R^i_{jk}(x, y) = \frac{\delta N^i{}_j}{\delta x^k} - \frac{\delta N^i{}_k}{\delta x^j}$ of the distribution N is given by

$$R^i_{jk}(x, y) = y^h R_h{}^i{}_{jk}(x) - (F^i{}_{j|k} - F^i{}_{k|j}), \tag{14}$$

where $R_h{}^i{}_{jk}(x)$ is the curvature tensor of Levi–Civita connection $\gamma^i_{jk}(x)$ and $F^i{}_{j|k}$ is the covariant derivation of tensor $F^i{}_j$ with respect to $\gamma^i_{jk}(x)$. From the previous formula it follows that $R^i_{jk}(x, y) = 0$ is equivalent to $R_h{}^i{}_{jk}(x) = 0$, $F^i{}_{j|k} = 0$. □

Now, we shall determine the canonical metrical connection $C\Gamma(N)$, [2, 4], of the space L^n.

Theorem 4. *The canonical metrical connection $C\Gamma(N)$ of the Lagrange space of electrodynamics has the following coefficients*

$$L^i_{jk}(x, y) = \gamma^i_{jk}(x), \quad C^i_{jk}(x, y) = 0. \tag{15}$$

Proof. The coefficients L^i_{jk}, C^i_{jk} of $C\Gamma(N)$ are expressed by the generalized Christoffel symbols:

$$L^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s}\right); \quad C^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^s}\right).$$

Taking into account the operators $\frac{\delta}{\delta x^i}$ from (12) and the fact that $\frac{\partial g_{ij}}{\partial y^k} = 0$, we get the announced result. □

The canonical metrical connection $C\Gamma(N)$ has the following torsion tensors

$$T^i_{jk} = S^i_{jk} = C^i_{jk} = P^i_{jk} = 0; \quad \text{and} \quad R^i_{jk} \text{ from (14).}$$

Also, it has the curvature tensors

$$R_h{}^i{}_{jk}(x), P_h{}^i{}_{jk} = S_h{}^i{}_{jk} = 0.$$

Therefore, we obtain, [2, 4]:

Theorem 5. 1°. *The Maxwell equations of the Lagrange space of electrodynamics, in the classical form, hold, i.e.*

$$F_{ij|k} + F_{jk|i} + F_{ki|j} = 0;$$

2°. *The Einstein equations of the Lagrange spaces of electrodynamics, in the classical form, hold good:*

$$R_{ij} - \frac{1}{2}R\gamma_{ij} = \kappa T_{ij}$$

and the conservation law $T^i{}_{j|i} = 0$ is satisfied.

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On a differential inequality I

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Abstract. We find conditions on the complex-valued function A defined in the unit disc U and the real constants α, β, such that the differential inequality

$$\Re[A(z)p^2(z) - \alpha(zp'(z))^2 + \beta] > 0$$

implies $\Re p(z) > 0$, where $p \in \mathcal{H}[1, n]$.

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1. Introduction and preliminaries

Denote by $\mathcal{H}[U]$ the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i [1, p. 35].

Lemma 1 (1, p. 35)**.** *Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be a function which satisfies*

$$\Re\psi(\rho i, \sigma; z) \leq 0,$$

where $\rho, \sigma \in \mathbb{R}$, $\sigma \leq -ds\frac{n}{2}(1 + \rho^2)$, $z \in U$ and $n \geq 1$.

If $p \in \mathcal{H}[1, n]$ and

$$\Re\psi(p(z), zp'(z); z) > 0$$

then

$$\Re p(z) > 0.$$

2. Main results

Theorem 2. *Let $\alpha \geq 0$, $\beta \leq ds\frac{\alpha n^2}{4}$ and let n be a positive integer. Suppose that the function $A : U \to \mathbb{C}$ satisfies*

$$\Re A(z) \geq -ds\frac{\alpha n^2}{2}. \tag{1}$$

If $p \in \mathcal{H}[1, n]$ and

$$\Re[A(z)p^2(z) - \alpha(zp'(z))^2 + \beta] > 0, \tag{2}$$

then

$$\Re p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\psi(p(z), zp'(z); z) = A(z)p^2(z) - \alpha(zp'(z))^2 + \beta. \tag{3}$$

From (2) we have

$$\mathbb{R}\psi(p(z), zp'(z); z) > 0, \text{ for } z \in U. \tag{4}$$

For $\sigma, \rho \in \mathbb{R}$ satisfying $\sigma \leq -\mathrm{d}s\frac{n}{2}(1+\rho^2)$, hence $-\sigma^2 \leq -\mathrm{d}s\frac{n^2}{4}(1+\rho^2)^2$, and $z \in U$, by using (1) we obtain

$$\begin{aligned}
\mathbb{R}\psi(\rho i, \sigma; z) &= \mathbb{R}[A(z)(\rho i)^2 - \alpha\sigma^2 + \beta] = \mathbb{R}(-\rho^2 A(z) - \alpha\sigma^2 + \beta) = \\
&= -\rho^2 \mathbb{R}\, A(z) - \alpha\sigma^2 + \beta \\
&\leq -\rho^2 \mathbb{R}\, A(z) - \frac{\alpha n^2}{4}(1+\rho^2)^2 + \beta \leq \\
&\leq -\rho^2 \mathbb{R}\, A(z) - \frac{\alpha n^2}{4} - \frac{\alpha n^2}{2}\rho^2 - \frac{\alpha n^2}{4}\rho^4 + \beta \leq \\
&\leq -\frac{\alpha n^2}{4}\rho^4 - \left[\mathbb{R}\, A(z) + \frac{\alpha n^2}{2}\right]\rho^2 + \beta - \frac{\alpha n^2}{4} \leq 0.
\end{aligned}$$

By using Lemma 1 we have that $\mathbb{R}p(z) > 0$. □

If $\beta = \mathrm{d}s\frac{\alpha n^2}{4}$, Theorem can be rewritten as follows

Corollary 3. *Let $\alpha \geq 0$, and let n be a positive integer. Suppose that the function $A : U \to \mathbb{C}$ satisfies*

$$\mathbb{R}\, A(z) \geq -\frac{\alpha n^2}{2}.$$

If $p \in \mathcal{H}[1, n]$ and

$$\mathbb{R}\left[A(z)p^2(z) - \alpha(zp'(z))^2 + \frac{\alpha n^2}{4}\right] > 0$$

then

$$\mathbb{R}p(z) > 0.$$

If $\alpha = 2$, $n = 1$, $A(z) = 1 + 2z$ from Corollary 3 we deduce

Example 4. *If $p \in \mathcal{H}[1, 1]$ then*

$$\mathbb{R}\left[(1+2z)p^2(z) - 2(zp'(z))^2 + \frac{1}{2}\right] > 0$$

implies

$$\mathbb{R}p(z) > 0.$$

If $\alpha = \mathrm{d}s\frac{1}{2}$, $n = 2$, $A(z) = 1 + 2z$ from Corollary 3 we deduce

Example 5. *If $p \in \mathcal{H}[1, 2]$ then*

$$\mathbb{R}\left[(1+2z)p^2(z) - \frac{1}{2}(zp'(z))^2 + \frac{1}{2}\right] > 0$$

implies $\mathbb{R}p(z) > 0$.

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Gray level image processing using algebraic structures

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Abstract. This paper presents a vector space structure defined on the interval $(0,1)$ that can be used in the gray level image processing area.

1. Introduction

The image enhancement is an important stage in the image processing domain. Usually, a gray level image is defined by a real and bounded function $f : \Omega \to V$, where $\Omega \subset R^2$ is the image support and V represents the gray level set. Sometimes, the images that must be processed have some suboptimal values for luminosity and contrast. Generally, the tuning of these parameters is made by using some gray level transform functions. One of the most known method for image enhancement is the histogram equalisation. The gray level transform is obtained by using the accumulated histogram. A much more flexible variant for the gray level transformation is that offered by the algebraic structures defined on real and bounded sets. Using of classical operations of real number algebra has the inconvenient that the obtained results can be found outside the permitted values, namely the gray level set V. In this paper, it is presented an algebraic structure having a logarithmic behaviour and being defined on the real and open interval $V = (0,1)$, which plays the role of the gray level space.

The first logarithmic image processing was made by Oppenheim [2] and Stockham [9] in the framework of homomorphic filters using as gray level space the set $(0,\infty)$. These are based on the multiplicative group structure of real and positive numbers [8]. Later, Jourlin and Pinoli [1] constructed a new logarithmic model using the set $(-\infty, M)$ with $M > 0$. In the framework of their model, Jourlin and Pinoli used the addition and the multiplication by positive scalars for image processing. The first logarithmic model defined on a real and bounded set $V = (-M, M)$ is that presented and used by Pătraşcu and Buzuloiu in [3, 4, 5, 6, 7].

The structure of this paper is the following: Section 2 presents a vector space structure defined on the set $V = (0,1)$; Section 3 presents the affine transform determination for image enhancement; Section 4 presents some experimental results and Section 5 comprises some conclusions.

2. The fundamentals of the logarithmic model

The most usual mathematical model for the gray level images is the real valued function defined on a bounded subset $\Omega \subset R^2$. Keeping in mind that the function value at a point (x,y) represents the luminosity of that pixel or reflectancy or transparency, it becomes clear that the functions we use are bounded ones (say, they take values in a bounded interval $[0,M]$). The problems appear when processing an image: the mathematical operations on real valued functions use implicitly the algebra of the real numbers i.e. on the whole real axis and we are faced with results that do not belong anymore to the interval $[0,M]$ - the only ones with physical meaning. Nevertheless this situation is generally accepted by using the truncation for the values out of the range $[0,M]$. The framework used in this paper, from this point of view, is radically different: namely, we want our set of the gray values to be a stable set with respect to the algebraic operations that we use - addition and scalar multiplication. In the set of gray levels $V = (0,1)$ we will define the addition $\oplus$ and the multiplication $\otimes$ by a real scalar and then, defining a scalar product $(\cdot|\cdot)_V$ and a norm $||\cdot||_V$, we shall obtain an Euclidean space of gray levels.

2.1. **The addition.** We define the sum of two gray levels, $v_{1\oplus}v_2$, by

$$v_1 \oplus v_2 = \frac{v_1 v_2}{(1-v_1)(1-v_2)+v_1 v_2}, \forall v_1, v_2 \in V \tag{1}$$

The neutral element for the addition is $\theta = 0.5$. Each element $v \in V$ has an opposite $w = 1 - v$. The addition $\oplus$ is stable, associative, and commutative. Thus, it follows that this operation induces on V a commutative group structure. We define the subtraction operation by

$$v_1 \ominus v_2 = \frac{v_1(1-v_2)}{(1-v_1)v_2+(1-v_2)v_1}, \forall v_1, v_2 \in V \tag{2}$$

Using the defined subtraction $\ominus$, we denote the opposite of v by $\ominus v$.

2.2. **The multiplication by a scalar.** We define the multiplication $\otimes$ of a gray level v by a scalar λ as

$$\lambda \otimes v = \frac{v^\lambda}{v^\lambda + (1-v)^\lambda}, \forall v \in V, \forall \lambda \in R \tag{3}$$

The above defined operations, the addition $\oplus$ and the scalar multiplication $\otimes$, induce on V a real vector space structure.

2.3. **The fundamental isomorphism.** The vector space of gray levels $(V, \oplus, \otimes)$ is isomorphic to the space of real numbers $(R, +, \cdot)$ by the function $\varphi : V \to R$, defined as

$$\varphi(v) = \frac{1}{4} \ln\left(\frac{v}{1-v}\right), \forall v \in V \tag{4}$$

The isomorphism φ satisfies

$$\varphi(v_1 \oplus v_2) = \varphi(v_1) + \varphi(v_2), \forall v_1, v_2, \in V \tag{5}$$

$$\varphi(\lambda \otimes v) = \lambda \cdot \varphi(v), \forall \lambda \in R, \forall v \in V \tag{6}$$

The particular nature of this isomorphism induces the logarithmic character of the mathematical model.

2.4. **The Euclidean space of gray levels.** The scalar product of two gray levels, $\langle \cdot | \cdot \rangle_V : V \times V \to V$ is defined with respect to the isomorphism from (4) as

$$\forall v_1, v_2, \in V, \ \langle v_1 | v_2 \rangle_V = \varphi(v_1) \cdot \varphi(v_2) \tag{7}$$

Based on the scalar product $\langle \cdot | \cdot \rangle_V$ the gray level space becomes an Euclidean space. The norm $|| \cdot ||_V : V \to [0, \infty)$ is defined via the scalar product

$$\forall v \in V, \ ||v||_V = \sqrt{\langle v | v \rangle_V} = |\varphi(v)| \tag{8}$$

3. Gray level image enhancement by affine transforms

Consider these affine transforms on the images set $F(\Omega, V)$, defined as follows: $\psi : F(\Omega, V) \to F(\Omega, V)$,

$$\psi(f) = \lambda \otimes (f \oplus \tau), \forall f \in F(\Omega, V), \tag{9}$$

where $\lambda \in R, \tau \in V$ while $\Omega \subset R^2$ is the image support. This form was preferred to be used because it shows that an image can be processed in two steps: a gray level translation with a constant value τ, which leads to a change in the image brightness, then a scalar multiplication by the factor λ - leading to a change in the image contrast. The parameters (λ, τ) were chosen in such a way as to get a new image very close (from a statistical point of view) to an image with an uniform gray level distribution on the set $V = (0, 1)$. This criterion shows the fact that the enhanced image must have its mean $\mu_u = 0.5$ and its variance $\sigma_u^2 = 1/12$. In fact, as a result, we are doing nothing else but to approximate the nonlinear transform yielded by the algorithm of gray level histogram equalization with an affine transform as the one in (9). In these conditions for any image f with the mean μ_f and the variance σ_f^2, the affine transform ψ becomes

$$\psi(f) = \frac{\sigma_u}{\sigma_f} \otimes (f \ominus \mu_f). \tag{10}$$

In the discrete case the mean μ_f and the variance σ_f^2 are defined by

$$\mu_f = \bigoplus_{(x,y)\in\Omega} (\frac{1}{card(\Omega)} \otimes f(x,y)) \tag{11}$$

$$\sigma_f^2 = \sum_{(x,y)\in\Omega} \frac{1}{card(\Omega)} ||f(x,y) \ominus \mu_f||_V^2 \tag{12}$$

where $card(\Omega)$ is the cardinality of the support Ω.

4. Experimental result

To exemplify, two images were picked out: "lax" in fig.1 (*left*), and "landsat" in fig.4 (*left*). The images are enhanced with the following affine transforms.
for image "lax",

$$\psi_1(v) = 4.62 \otimes (v \ominus 0.49) \tag{13}$$

and for image "landsat",

$$\psi_2(v) = 5.31 \otimes (v \ominus 0.21) \tag{14}$$

The graphics of the affine transforms are shown in fig.2 and fig.5. The enhanced images can be seen in fig.3 and fig.6 (*left*). The gray level histograms can be seen in fig.1 and fig.4 (*right*) for the original images and in fig.3 and fig.6 (*right*) for the enhanced images.

5. Conclusions

It was presented a mathematical model for the gray level images by defining an algebraic structure on the bounded interval $(0, 1)$, and by introducing some basic operations (addition, scalar multiplication) and functions (scalar product, norm). This structure, based on a logarithmic model, provides gray level operations, which yield results that are always confined to the underlying bounded interval of allowed values. It is proposed a fully automatic image enhancement method based on the use of an affine transform. The parameters of the affine transform are computed by approximating the classical histogram equalization technique. The tests show that the proposed techniques allow the automatic correction of the illumination problems like histogram equalization. The proposed method is just another strong argument for the rich potential of the logarithmic image processing models.

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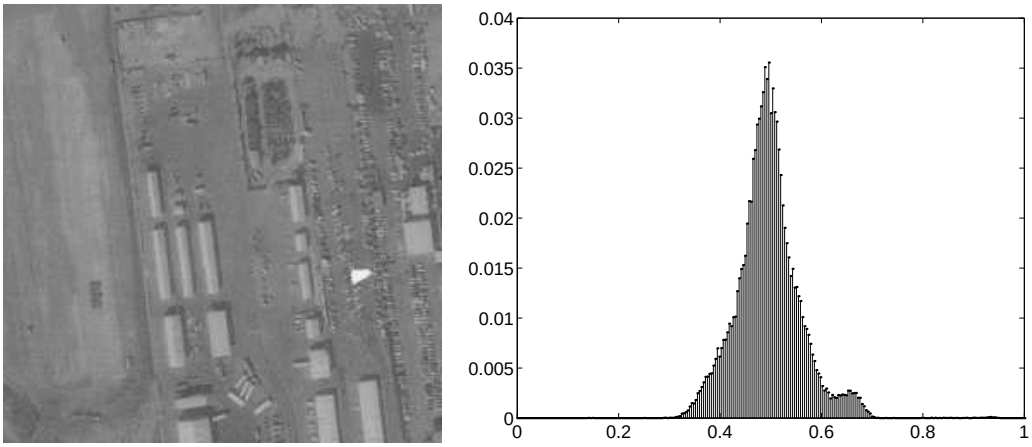

FIGURE 1. The original image lax (*left*) and the histogram (*right*).

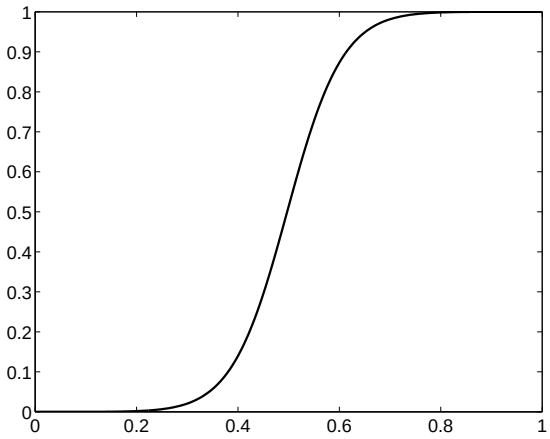

FIGURE 2. The affine transform for image "lax".

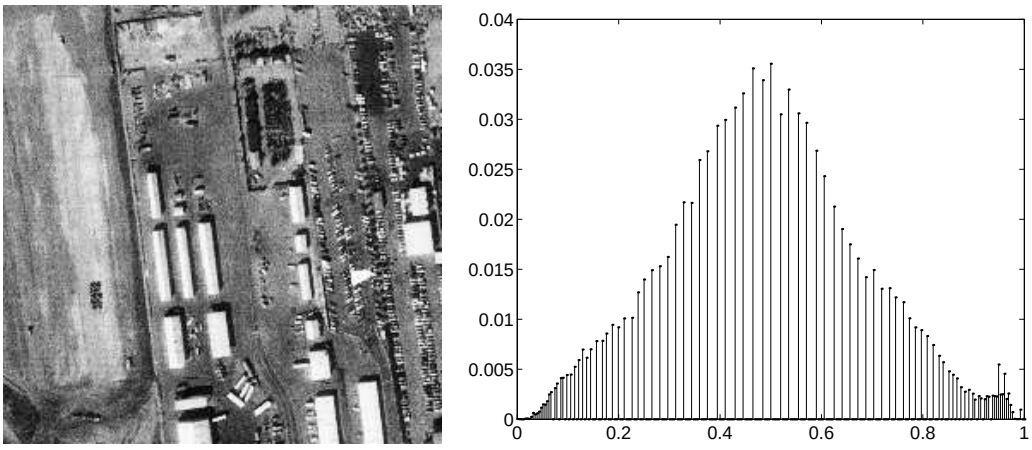

FIGURE 3. The enhanced image lax (*left*) and the histogram (*right*).

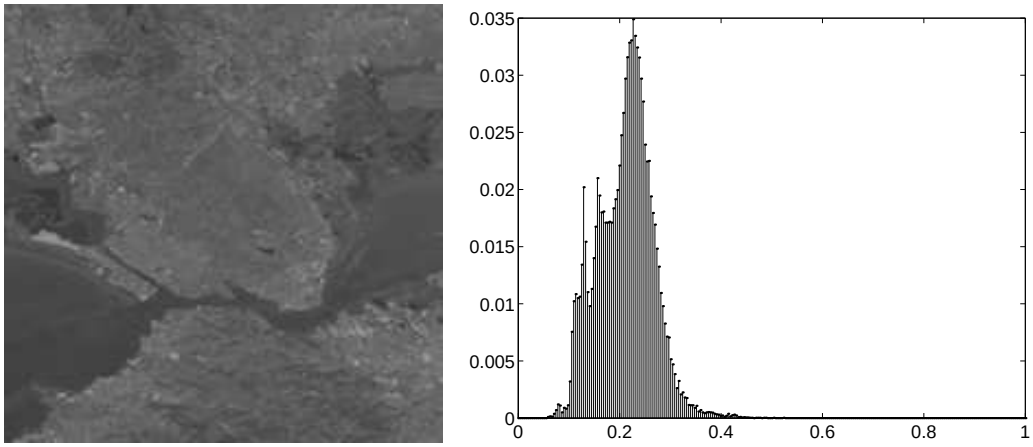

FIGURE 4. The original image landsat (*left*) and the histogram (*right*).

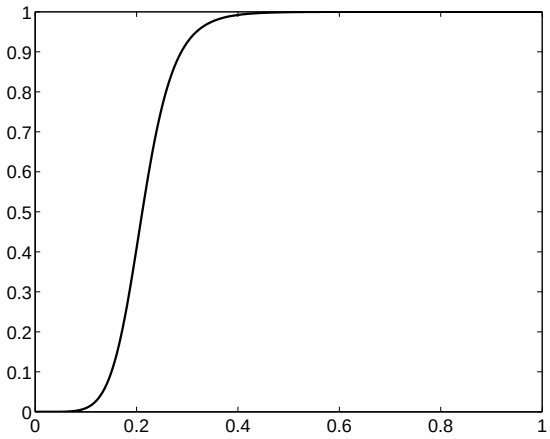

FIGURE 5. The affine transform for image "landsat".

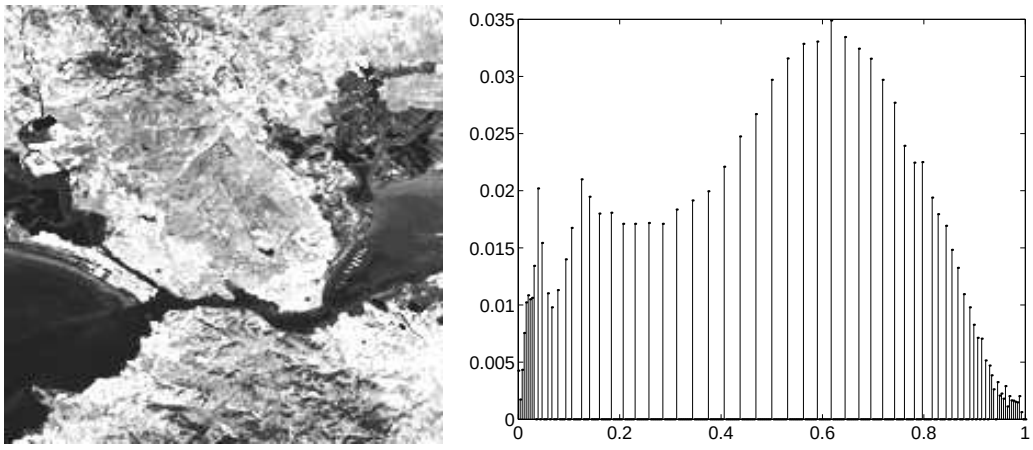

FIGURE 6. The enhanced image landsat (*left*) and the histogram (*right*).

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Using data mining techniques for increasing security in email system Internet-based

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Abstract. As the Internet grows at a phenomenal rate, email systems have became a widely used electronic form of communication. Everyday, a large number of people exchange messages in this fact and inexpensive way. With the excitement on electronic commerce growing, the usage of email will increase more exponential.
In this paper we explore a data mining capability which involves mining techniques in email system Internet-based for increasing security.

Keywords: data mining, text mining, pattern discovery, email message

1. Introduction

The rapid progress of computer and network technologies makes it easy to collect and store a large amount of semi-structured texts such as webpages, HTML/XML archives and e-mails. These text data can be thought of as large scale text databases, and thus it becomes important to develop an efficient tool to discover interesting security knowledge from such text databases.

Data mining refers to process of non-trivial extraction of implicit, previously unknown, and potentially useful security information from large databases.

In this paper we explore a data mining capability which involves mining techniques in email system Internet-based for increasing security.

Solving Internet security problems greatly assist surveillance intelligence activities. For example, the discovery of user account communities and the discovery and detection of certain community behavior patterns can be directed to uncover certain classes of covert, clandestine or terrorist behavior performed with Internet resources.

However, it is difficult to directly apply the data mining technologies to such text or semi-structured data since these text databases consist of:

(i) heterogeneous and,

(ii) huge collections of

(iii) un-structured or semi-structured data.

In a broad sense, e-mail messages are semi-structured documents that possess a set of structured fields with predefined semantics and a number of variable length free-text fields [1]. In a formal way, such a document can be represented as fig.1.

Field 1 to *Field s* are structured fields and usually contain information about document, such as authorship, date, organization, layout of the text body etc. As the major content of the document, *Field s+1* to *Field s+t* are variable length free-text fields, such as subject area, abstract, the body and references. While most classification work focuses either on the structured part or the text part, we argue that both the structured fields and the free text portion could contain important information for increasing security in e-mail system Internet-based.

Our research goal is to devise an efficient semi-automatic tool for increasing email security, and that supports human discovery from large e-mail databases (mailing lists). Therefore, we require a fast pattern discovery algorithm that can work in time to respond in real time on an semi-structured data set of total size n. To achieve this goal, we adopt the framework of *optimized pattern discovery* [3] described below, and develop efficient and robust pattern discovery algorithms combining the advanced technologies in string algorithm, computational learning theory, and data mining.

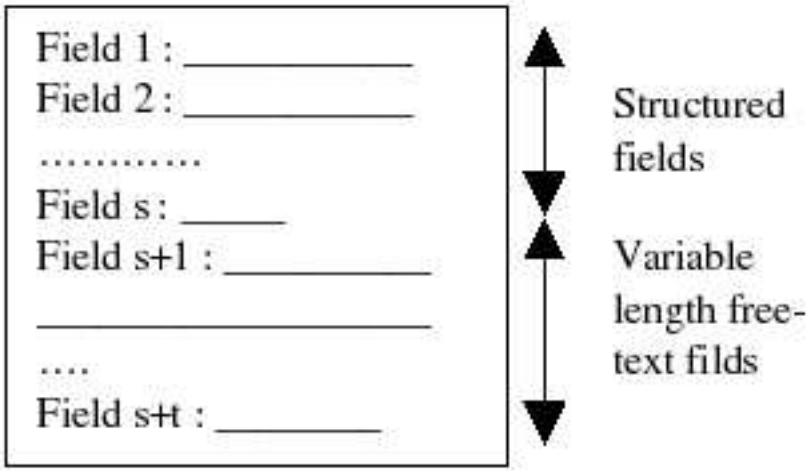

FIGURE 1. Modeling email message.

1.1. **Optimized pattern discovery in email-documents.** The framework of *optimized pattern discovery*, adopted in this paper, is originally proposed by Arimura et al. [2,3] in the field of data mining and also known as *Agnostic PAC learning* in computational learning theory. In optimized pattern discovery, a pattern discovery algorithm tries to find a pattern from a given hypothesis space that optimizes a statistical measure function, such as *classification error*, *information entropy*, *Gini index*, and χ^2 *index* to discriminate a given target (or *positive*) data set from another background *(negative*) data set [2].

More precisely, we define the optimized pattern discovery as follows. Suppose that we are given a set $S = \{s_1, \ldots, s_m\}$ of texts and an objective function $F : S \to \{0, 1\}$, where each s_i is called an *email-document*. The value of the objective function $F(s_i)$ indicates that the document s_i is interesting (positive) if $F(s_i) = 1$ and not interesting (negative) otherwise.

Let P be a (possibly infinite) class of *patterns*, where for any pattern $p \in P$ and any string s, we define $p(s) = 1$ if p *matches* s and $p(s) = 0$ otherwise.

Let S be a set of email-documents and F be an objective function. Then, a pattern p defines a contingency table *(M1;M0;N1;N0),* where $N1$ (resp., $N0$) is the number of all positive (resp., negative) email-documents in S, and $M1$ (resp., $M0$) is the number of all positive (resp., negative) documents $s \in S$ such that $p(s) = 1$.

An *impurity function* is any real-valued function I*: [0; 1]* $\to R$ such that:

(i) it takes the maximum value $I(1/2)$;

(ii) the minimum value $I(0) = I(1) = 0$;

(iii) is convex, i.e. $I((x + y)/2) \geq (I(x) + I(y))/2$ for every $x, y \in [0; 1]$.

The followings are examples of impurity functions:

- the prediction error: $I_1(x) = min(x, 1- x)$;
- the information entropy $I_2(x) = -x \log x - (1- x)\log(1- x)$;
- the Gini index: $I_3(x) = 2x(1- x)$.

Then, the evaluation function based on I over S and F is

$G^I_{S,F}(p) = I(M1/N1)N1 + I(M0/N0)N0$,

where *(M1;M0;N1;N0)* is the contingency table defined by the pattern p over S and F.

Now, we state the our data mining problem, called the *optimal pattern discovery problem*, as follows. Let P be the class of candidate patterns and let be any impurity function.

1.2. **Optimal Pattern Discovery Problem.** *Given:* a set S of email-documents and an objective function $F : S \rightarrow \{0,1\}$.

Problem: Find an optimal pattern $p \in P$ that minimizes the cost $G^I_{S,F}(p)$ within P.

In what follows, we consider the information entropy measure only, but not limited to it. From recent development in learning theory, it is known that any algorithm that efficiently solves, e.g. classification error minimization, can approximate arbitrary unknown probability distributions and thus can work with noisy environments [3].

An intuition behind the application of the optimized pattern discovery to text mining can be explained as follows. Suppose that we are given as the target set a collection of email-documents, "*irakian war news mailing list*" (over 2148 subscribers) for years of 2002-2003. We want to take a look at the contents and find a set of topic keywords characterizing the major topics in war information arising for years of 2002-2003.

A possible way to find such keywords or phrases is to find the keywords that frequently appear in the target set as in classical data mining. However, this does not works in most text collections because in a typical English text, the most frequent keywords are *stopwords* like "the" or "an" (Table 1 (a)). These keywords are basic constituents of English grammars and convey no information on the contents of the text collection. Such frequent but less informative stopwords may hide less frequent informative keywords. The traditional information retrieval data mining technique called *stopword elimination* may not work, too.

A basic idea behind our method is to use an average set of texts as the control set used for canceling the occurrences of frequent and non-informative keywords. The control set will be a set of email-documents randomly drawn from the whole email collection or the internet. We can easily observe that most stopwords appear evenly in the target and the control set, while informative keywords appear more frequently in the target set than the control set. Therefore, the optimized pattern discovery algorithm will find those keywords or phrases that appear more frequently in the target set than the control set by minimizing a given statistical measure such as the information entropy or the prediction error (Table 1 (b)-(c)).

The class of patterns we consider is the class of proximity phrase association patterns [3]. By a *phrase* we mean any string of tokens, which may be either letters or words, of arbitrary length.

A *phrase association pattern* (phrase pattern) is an expression of the form (<attack>, <irakian military bases>; 8) which expresses that phrase <attack> first appears in a mailing list and then phrase <irakian military bases> follows within eight words. A phrase pattern can contain arbitrary many but bounded number of phrases as its components. If the order of the phrases in a pattern matters as in the above example then we call it *ordered* and otherwise *unordered*. Proximity phrase association patterns can be regarded as a generalization of association rules in transaction databases [2] such that:

(i) each item is a phrase of arbitrary length;

(ii) items are ordered; and

(iii) a proximity constraint is introduced.

2. A FAST AND ROBUST TEXT MINING ALGORITHM FOR ORDERED PATTERNS

If the maximum number of phrases in a pattern is bounded by a constant d then the frequent pattern problems for both unordered and ordered proximity phrase association patterns are solvable by *Enumerate-Scan* algorithm [3], a modification of a naive generate-and-test algorithm, in $O(n^{d+1})$ time and $O(n^d)$ scans; although it is still too slow to apply real world problems.

Adopting the framework of optimized pattern discovery, we have modified and developed the efficient *Split-Merge* algorithm, that finds all the optimal patterns for the class of *ordered k-proximity d-phrase* association patterns for various measures including the classification error and information entropy [3, 4]. The algorithm quickly searches the hypothesis space using dynamic reconstruction of the content index, called a *suffix array* with combining several techniques from computational geometry, string algorithms and data mining.

We showed that the Split-Merge algorithm runs in *almost linear time in average*, more precisely in $O(k^{d-1}N(\log N)^{d+1})$ time using $O(k^{d-1}N)$ space for nearly random texts of size N [4].

3. Data sets and experiments

We applied our email mining method to *interactive email-document mailing list.* Based on the Split-Merge algorithm [4], we developed a prototype system on a Microsoft NT workstation, and run experiments on a medium sized English email collection.

We used an English mailing list collection, called "*irakian war news mailing list*", which consists of emails of 8.4MB from November 2002 to April 2003. The sample set is a collection of semi-structured email texts of the total size 10.2MB obtained from "irakian war news mailing list" by removing all but category tags. The target (positive) set consists of 620 emails with category bases and the background (negative) set consists of 7244 emails with other categories such as war, military, target, missile, and so on. The average length of emails is 1120 letters.

By experiments on "irakian war mailing list" above of 10.2MB, the prototype system finds the best 300 patterns at the entropy measure in seconds for d = 2 and a few minutes for d = 4 and with k = 2 words using a few hundreds mega-bytes of main memory on Intel PIII, 500MHz, c++ on Windows NT, 512MB main memory.

4. Results and conclusion

In Table 1, we show the list of the phrase patterns discovered by our mining system, which capture the category bases relative to other categories of "irakian war news". In Fig. 1 (a), we show the list of most frequent keywords discovered by traditional frequency maximization method. On the other hand, we list in Fig. 1 (b) and (c), we show the list of optimal patterns discovered by entropy minimization method. The patterns of smallest rank (1-8) contain the topic keywords in the major email-documents (Table 1 (b)). Such keywords are hard to find by traditional frequent pattern discovery because of the existence of the high frequency words such as <the> and <are>. The patterns of medium rank (311-318) are long phrases, such that <intelligence agencies> and <irakian oil platform>, as a summary (Table 1 (c)), which cannot be represented by any combination of non-contiguous keywords.

(a) Frequency maximization	(b) Entropy minimization	(c) Entropy minimization
1 <news>	1 <war>	311 < war began>
2 <the>	2 <military>	312 <irakian oil platform>
3 <to>	3 <bases>	313 <intelligence agencies>
4 <of>	4 <target>	314 <military port>
5 <and>	5<irakian>	315 <civilian victims>
6 <in>	6 <attack>	316 <democracy>
7 <all>	7 <forces>	317 <water>
8 <a>	8 <missile>	318 <freedom>

Table 1. Phrase mining with entropy optimization to capture bases category. To see the effectiveness of entropy optimization, we mine only patterns with d=1, i.e., single phrases.

(1) The best 8 frequent phrases found by traditional frequent pattern mining.
(2) Short phrases of smallest rank, 1-8.
(3) Long phrases of middle rank, 311-318 found by entropy minimization mining.

The data set consists of 9453 email-documents of 8.4 MB from mailing list in 2002-2003.

Table 2 shows an experiment on interactive email-documents browsing, where we try to find an article containing a specific topic from a collection of emails by using optimized pattern discovery combined with keyword search.

(a) Stage 1	(b) Stage 2	(c) Stage 3
Rank Pattern	Rank Pattern	Rank Pattern
30 <navy>	1 <military>	1 <troops> <and> <the>
31 <military>	2 <u.s.>	2 <troops> <on> <the>
32 <tank>	**3 <troops>**	3 <troops> <were> <action>
33 <desert>	4 <missiles>	**4 <troops> <were><still in action>**
34 <attack>	5 <sea>	5 <troops> <to> <after>
35 <subunit>	6 <tornado>	6 <troops> <on> <action>
36 <water>	7 <machinegun>	7 <troops> <said> <were>
37 <gulf>	8 <gulf>	8 <troops> <commander> <of>

TABLE 2. Email-documents browsing by optimized pattern discovery: (a) First, a user tries to mine the original target set using optimized pattern discovery over phrases using the background set (d = 1; k = 0). The user selected a term <subunit> of rank 35. (b) Next, the user mines a subset of articles relating to <subunit> and mine this set again by optimized phrases (d = 1; k = 0). He obtained topic terms on troops. (c) Finally, the user tries to discover optimized patterns starting with troops on the same target set (d = 4; k = 2 words). The underlined pattern indicates an action by a subunit of troops.

First, we suppose that a user is looking for articles related to the war problem, but he does not know any specific keywords enough to identify such articles:

(a) Starting with the original target and the background sets related to subunit category in the last section, the user first finds topic keywords in the original target set using optimized pattern mining with d = 1, i.e. phrase mining. Let troops be a keyword found.

(b) Then, he builds a new target set by drawing all articles with keyword troops from the original target set. The last target set is used as the new background set. As a result, we obtained a list of topic phrases concerned to troops.

(c) Using a term troops found in the last stage, we try to find long patterns consisting of four phrases such that the first phrase is troops using proximity k = 2words. In the table, a pattern <troops> <were> <still in action> found by the algorithm indicates that there is an action by a subunit of troops.

In conclusion, in this paper we investigate and experiment a new method for large emails database on internet-based text mining used for increasing security in email system.

First, we formalized text mining problem as the optimized pattern discovery using a statistical measure. Then, we gave a fast and robust pattern discovery algorithm which was applicable to a large email collection and present the experiments and the resultants.

Thus, it will be an interesting future problem to develop an efficient algorithm to find optimal pattern in emails collection for increasing security.

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Improving the Xia-You Group signature scheme

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Abstract. A group signature scheme allows any group member to sign on behalf of the group in an anonymous and unlinkable fashion. In case of dispute, group manager can reveal the identity of the signer. Recently, S. Xia and J. You proposed a group signature scheme based on identity with strong separability in which the revocation manager can work without the involvement of the membership manager. But their group signature scheme is not secure because two or more group members can collude to construct a valid signature and any group member can forge a valid membership certification. In this paper we improve the Xia-You group signature scheme such that our group signature scheme is secure against coallition attack.

1. Introduction

Digital signature can be used to authenticate the identity of the sender of a message or the signer of a document and to identify data integrity. With high Internet development, digital signature plays an important role in electronic commerce and identity authentication. Group signature is very important signature with additional functionality.

Group signature is a relatively new concept introduced by Chaum and Heijst [4] in 1991. A group signature allows the signer to demonstrate knowledge of a secret with respect to a special document. A group signature is publicly verifiable: it can be validated by anyone in possession of a group public key. However, group signatures are anonymous in that no one, with the exception of a designated group manager, can determine the identity of the signer. Furthermore, group signatures are unlinkable which makes computationally hard to establish whether or not multiple signatures are produced by the same group member. In exceptional cases any group signature can be opened by a group manager to reveal unambiguously the identity of the actual signer. At the same time, no one, including the group manager, can misattribute a valid group signature. Many group signature schemes have been proposed [1, 2, 3, 6]. However all of them are much less efficient than regular signature schemes (such as DSA or RSA). Designing an efficient group signature scheme is still an open research problem. Recently, several group signatures based on identity [9, 10] was proposed. Unfortunately, these schemes have been attacked soon. In [10], Xia-You presented a novel group signature scheme based on identity. In [5], it is analyzed the security of Xia-You group signature scheme and two or more group members can collude to construct a valid signature and any group member can forge a valid membership certificate. All the above attacks cannot be traced by the group manager.

2. Xia-You's Group Signature Scheme

Xia-You [10] presented a novel group signature scheme with strong separability based on the identity cryptographic system, firstly introduced by Shamir [8]. More detailed description refers to the original paper [10].

2.1. **Trusted Authority Setup.** The trusted authority chooses two big prime numbers $p_1 \equiv \pm 1 \pmod 8$, $p_2 \equiv \pm 3 \pmod 8$ of about 100 decimal digits such that $p_1 - 1$ and $p_2 - 1$ contains several prime factors of $13 \sim 15$ decimal digits, but no larger on, and that $(p_1 - 1)/2$ and $(p_2 - 1)/2$ are relatively prime. Let $n_1 = p_1 * p_2$. According to the selection of p_1 and p_2, the Jacobi symbol $\frac{2}{n_1} = -1$ and the trusted authority can easily find the discrete logarithm modulo p_1 and p_2 respectively. Finally, the trusted authority randomly chooses a number g which meets $g < \min(p_1, p_2)$ and publishes (n_1, g) and keeps (p_1, p_2) secret.

2.2. **Group Member's Private Key Generation.** Suppose Alice wants to join a group. Alice submits her own identity information D_A to the trusted authority and the trusted authority sets $ID_A = D_A \pmod{n_1}$ if $D_A/n_1 = 1$ or $ID_A = 2D_A \pmod{n_1}$ if $D_A/n_1 = -1$. Finally, the trusted authority computes the private key x_A of Alice as follows

$$ID_A = g^{x_A} \pmod{n_1}.$$

2.3. **Group Manager's Setup.** The group manager chooses two strong primes p_3, p_4 and computes a RSA modulo number $n_2 = p_3 * p_4$ $(n_1 < n_2)$. The public exponent is e and the private exponent is d. The group manager chooses two integers $x \in \mathbb{Z}_{n_1}, h \in \mathbb{Z}_{n_1}^*$ and computes $y = h^x \pmod{n_1}$ satisfying $y \in \mathbb{Z}_{n_1}^*$. Let $H(\cdot)$ be a hash function such that $H : \{0,1\}^* \to \mathbb{Z}_{n_1}$. The public key of the group manager is (n_2, e, h, y, H) and his secret key is (x, d, p_3, p_4).

2.4. **Signing Phase.** When Alice wants to join the group, the group manager computes $z_A = ID_A^d \pmod{n_2}$ and sends it to Alice in a secure way. Alice verifies the validity of z_A by $ID_A = z_A^e \pmod{n_2}$. When a group member Alice, with (x_A, z_A), signs a message m, she chooses random integers $\alpha, \beta, \theta, \omega \in \mathbb{Z}_{n_1}$ and $\delta \in \mathbb{Z}_{n_2}$ and computes as follows:

$$\begin{aligned}
A &= (y^\alpha z_A) \pmod{n_2} \\
B &= y^\omega ID_A \\
C &= h^\omega \pmod{n_1} \\
D &= H\left(y\|g\|h\|B\|\widehat{B}\|C\|v\|t_1\|t_2\|t_3\|m\right) \\
E &= \delta - D(\alpha e - \omega) \\
F &= \beta - D\omega \\
G &= \theta - Dx_A
\end{aligned}$$

where $\widehat{B} = B \pmod{n_1}$, $v = (A^e/B) \pmod{n_2}$, $t_1 = y^\alpha \pmod{n_2}$, $t_2 = \left(y^\beta g^\theta\right) \pmod{n_1}$, $t_3 = h^\beta \pmod{n_1}$. Finally, the group signature on m is (A, B, C, D, E, F, G).

2.5. **Verification Phase.** If the verifier validates the message-signature pair $\{m, (A, B, C, D, E, F, G)\}$, verifier computes: $\widehat{B}' = B \pmod{n_1}$, $v' = (A^e/B) \pmod{n_2}$, $t_1' = \left(v'^D y^E\right) \pmod{n_1}$, $t_2' = \left(\widehat{B}'^D y^F g^G\right) \pmod{n_1}$, $t_3' = \left(C^D h^F\right) \pmod{n_1}$, $D' = H(y\|g\|h\|B\|\widehat{B}'\|C\|\ v'\|t_1'\|t_2'\|t_3'\|m)$ iff $D' = D$, the verifier accepts the signature.

3. Improved Xia-You Group Signature Scheme

In this section we present the improved Xia-You group signature scheme such that this scheme is secure against coalition attacks. In this description, the group manager is also a trusted authority.

3.1. **Setup.** The setup procedure of our group signature scheme is as follow. The group manager executes next steps:

(1) Chooses two big prime numbers $p_1 \equiv \pm 1 \pmod 8$, $p_2 \equiv \pm 3 \pmod 8$ of about 100 decimal digits such that $p_1 - 1$ and $p_2 - 1$ contains several prime factors of $13 \sim 15$ decimal digits, but no larger on, and that $(p_1 - 1)/2$ and $(p_2 - 1)/2$ are relatively prime. Let $n = p_1 * p_2$.

(2) According to the selection of p_1 and p_2, the Jacobi symbol $\frac{2}{n} = -1$ and the group manager can easily find the discrete logarithm modulo p_1 and p_2 respectively.

(3) Chooses two integers $x \in \mathbb{Z}_n, h \in \mathbb{Z}_n^*$ and computes $y = h^x \pmod n$ satisfying $y \in \mathbb{Z}_n^*$.
(4) Let $H(\cdot)$ be a hash function such that $H : \{0,1\}^* \to \mathbb{Z}_n$.
(5) The public key of the group manager is (n, h, y, H) and his secret key is (x, p_1, p_2).

3.2. **Join.** Suppose now that Alice wants to join the group. We assume that communication between the group manager and Alice is secure, i.e., private and authentic. To obtain her membership certificate, Alice must perform the following protocol with the group manager:

(1) let ID_A be a string denoting the identity of a user Alice. Then, Alice sends ID_A to the group manager;
(2) the group manager computes $z_A = ID_A^x \pmod n$ and sends it to Alice in a secure way.

3.3. **Sign.** In our scheme, ID_A is the public component of a RSA signature public and private key pair generated by Alice herself. The public and private key pair will be referred to as (ID_A, d_A) in the remainder of this paper. First, the user Alice signs a message $m \in \{0,1\}^*$ with her private key d_A and the corresponding RSA signature scheme

$$RSA\,\mathrm{Si}\,g = m^{d_A} \pmod n.$$

Then, the group member Alice can generate anonymous and unlinkable group signatures on a message $m \in \{0,1\}^*$ as follows:

(1) chooses a random number $k \in \mathbb{Z}_n^*$;
(2) computes $S = H\left(m \parallel y \parallel h \parallel z_A^k\right)$;
(3) the group signature $Group\,\mathrm{Si}\,g$ is then the concatenation of the previously generated RSA signature $RSA\,\mathrm{Si}\,g, S$ with the Alice's public key ID_A:

$$Group\,\mathrm{Si}\,g = m^{d_A} \pmod n \parallel S \parallel ID_A.$$

3.4. **Verify.** A user verifies that the signature was generated by Alice and not by the group manager by verifying using the Alice's public key ID_A and the corresponding RSA signature that $RSA\,\mathrm{Si}\,g$ is valid

$$m = RSA\,\mathrm{Si}\,g^{ID_A} \pmod n.$$

Since the group manager does not know the private key d_A it will not be able to generate a valid $RSA\,\mathrm{Si}\,g$.

3.5. **Open.** The group manager knows the identity ID_A of the user Alice that is associated with it. This binding is established during the **Join** phase. As a result, it is easy for a group manager, given a message $m \in \{0,1\}^*$ and a valid group signature $Group\,\mathrm{Si}\,g$, to determine the identity of the signer.

4. Security Considerations

In this section, we access the security of our group signature scheme according to the security properties defined in [1]. In our scheme, a group signature is the concatenation of the identity based signature with the user's public key. Therefore if the underlying identity RSA-based signature provides anonymity and if the user's public key does not reveal any information about the user, anonymity is guaranteed by the group signature scheme. In our group signature scheme, a group member cannot sign behalf of other members because it does not know the other members' private keys. The group manager knows each users' private key z_A, but he do not knows the users' RSA private key d_A. Since, the group manager generates each member private keys from their public keys, it can easily identify the actual signer of a valid group signature by looking at the public key exponent in the group signature $Group\,\mathrm{Si}\,g$. A colluding subset of users, that have received their private key from the group manager, cannot generate a valid signature that the group manager cannot link to one of the colluding users.

5. Conclusion

This paper presented an improved Xia-You group signature scheme which is secure against the coalition attack. The generated group signature can handle large groups since the group public key and parameters are constant and do not depend on the group members. The security of such a group signature depends on the security of the RSA based signature scheme it was derived from. In our group signature scheme we used the ideas from reference [7].

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An algorithm for statistical data analysis

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1. Introduction

Recently, it often takes unrealistic time to obtain a result when we analyze a lot of data, for example in the case of data mining. Then, one of ways to solve this time-consuming problem is to use Parallel Virtual Machine (PVM), which allows us to compute quickly such as a super computer, whereas it requires much labor to use. In this paper, we would like to propose a Graphical User Interface (GUI) for parallel data analysis with PVM.

In data analysis field, we have many optimization problems that require computation for different initial values to search optimum. We even reach to some local optimum with one initial value, we need to restart from another initial value to search better local optimum. We need to search the local optimum repeatedly, starting with many different initial values, finally to find the global optimum, that requires much computational power. In the paper, a system of parallel computation for such problems with PVM library is presented.

2. Problems of PVM

PVM is a software library which enables many computer machines to work as a single high performance machine. Computers in PVM environment should be TCP/IP connected. PVM allows machines with different Operating System connected, for example, with Linux and Windows NT. It is useful for parallel data analysis, but it has some operational problems. Because we operate PVM system with command line, we must type a PVM command with various options, for example to redirect output of a job to console. Moreover, PVM doesn't support command history. Consequently, we need to type the same PVM command repeatedly. If we use PVM in a large amount of machines, listing hostnames of all machines must be a time-consuming operation. When we finish programming and preparing to execute it, we still have business to do. Another problem is to copy an executable file to all hosts, which is inefficient to copy it manually.

PVM is a software package that was developed by Oak Ridge National Laboratory in the USA. It allows a collection of computers hooked together by a network to be used as a single large parallel computer and, realizes parallel computing with a mechanism, called " message passing" . With PVM it is possible to develop some statistical data analysis programs, such as clustering or Sliced Inverse Regression with projection pursuit.

3. Proposed GUI for PVM

To overcome these problems, we developed a GUI system for PVM. This system has many functions, including as follows:

- spawn tasks easily;
- make a hostfile semi-automatically;
- copy an executable file to all hosts with using hostfile.

It is useful to show a snapshot of such a GUI system.

4. Optimization for Data Analysis

Optimization problems are in general, formulated as follows:

$$\text{minimize } f(x) = f(x_1, x_2, \ldots, x_n) \to R1, \text{ subject to } x\, D\, Rn.$$

We start with the initial point $x_0 = (x_{0_1}, x_{0_2}, \ldots, x_{0_n})$. With optimization algorithm we get reached to minimum, local minimum, that depends on the initial point

$$f(x) = f(x_1, x_2, \ldots, x_n) \to \text{ min (local)}.$$

We need to search local minima repeatedly starting with various initial points to get the "global" minimum:

$$\text{min (global)} = \text{minimize } (\text{ min/local}_1,\ \text{min/local}_2, \ldots).$$

With PVM master-slave ides, master machine delivers initial points to each slave machine on which local minimization is executed. After receiving local minimum from a slave, master sends another initial point to the slave if waiting queue of initial points at the master is not empty. Master machine selects the smallest value among local minima after receiving all local minima from slave machines. We regard the value as the global minimum.

We implement k-means clustering algorithm, where we need to minimize value of within-group dispersion. The algorithm is applied to data of 3000 points in a plane to be clusterized, starting with initial clustering. We get local minimum value of within-group dispersion starting with a initial clustering, then with another initial clustering we get next local minimum.

5. Creating a Metadata Repository

It is important for each biostatistics department to create a centralized networked clinical trials database system. The Biostatistics Information Tracking System (BITS) contains over 1000 clinical trials conducted in support of finding the causes, prevention and cure of a disease. BITS was originally developed using the Advanced Revelation software system, deployed over a PC network. The system is password protected to restrict access to research staff and investigators.. BITS incorporates all data required for protocol administration, survival analysis, and employs electronic interfaces to demographic and laboratory data.

For all patients registered onto a clinical trial, data on eligibility status, informed consent date, diagnosis, study arm, dates on and off treatment, follow-up interval, last contact date, relapse and survival status are computerized in BITS. Protocol – specific data include the prior treatment, protocol treatment, toxicity, treatment response, and any other results required for the final analysis. At the time of a full study analysis the data are exported from BITS into SAS for programming by the department's biostatisticians. The system can be migrated to MS SQL Server with Web-based screens and acannnable forms as the new interface applications and in this case it is very useful to use the PVM software, as well as the above mentioned algorithm.

A critical lesson learned during the construction and subsequent development of BITS is that the metadata are key to a sound, effective data system. Metadata are " data about the data" , that is information concerning the type and meaning of the data that stored in an electronic system. The metadata consist of two components, the " business directory" that provides a definition of each data element, along with the key words, synonyms, and directives for collecting each element and, the " technical directory" which includes information obtained directly from the data model itself. It is necessary to create the electronic storage system for this information in the form of a " metadata repository" . The steps in planning this one included: writing of the project specification and gathering user requirements; analysis of data dictionary requirements; creation of the metadata repository model; validation of this model; construction of the physical database; conducting an export of existing metadata from BITS into the metadata repository. And updating or inserting additional business and technical data as need.

The next step is to create the design specification for the Web application front end to edit and access the data stored in the metadata repository. A critical step which is often overlooked is the creation and

routine implementation of Standard Operating Procedures (SOP) for on-going timely maintenance of the metadata repository. After populating the metadata repository with the BITS metadata elements, we desired to utilize this system to harmonize and align data elements across two additional data systems that have been developed by and/or are maintained within a department of biostatistics.

This system will facilitate the training and quality assurance of the data being collected for statistical data analysis. Another goal is to expand and utilize this metadata repository approach to manage data contained in a research data warehouse. This will represent the optimal means to facilitate future complex and multi-disciplinary analyses.

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Maxima and minima problems by using MAPLE V

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Abstract. This paper shows how MAPLE V could be used to locate and test global and local extrema and also to solve optimisation problems that have constraints. These problems are studied by the students in the first year of technical faculties and can be taught even if the students lack advanced programming knowledge.
We present here some programs made in MAPLE V which can be used in fast solving of different exercises of extrema and constrained optimisation problems.

1. Introduction

In many fields researchers find MAPLE V to be an essential tool for their work. MAPLE V is ideal for formulating, solving and exploring mathematical models.

Instructors use it to present lectures. Educators in high schools, colleges and universities have revitalized traditional curricula by introducing problems and exercises that exploit MAPLE V's interactive mathematics; students can concentrate on important concepts rather than tedious algebraic manipulations.

The way in which we use MAPLE V is in some aspects personal and depends on our needs.

When we work on a problem in a traditional manner, attempting a particular method of solution, it may take hours and many pages of paper. MAPLE V allows us to tackle with much larger problems and frees us from human errors.

The purpose of the paper is to show how MAPLE V could be used to locate and test global and local extrema and also to solve optimization problems that have constraints.

The programms that are presented here consist in step by step instructions. This allows us to easily modify a step or insert a new one in our solution method. MAPLE V can then compute the new result easily.

2. Local and global extrema

2.1. **Statement of the problem.** Let f be a function, $f : A \subset R^k \to R \ \ (k = 2, 3)$. A point a from A is said to be "a relative minimum (or maximum)" if there is a neighborhood of the point a such as $f(x) \geq f(a)$ $(f(x) \leq f(a))$ for every x which belong to that neighborhood.

To find relative maxima, minima and saddle points of a function of two or three variables we use the following procedure (we assume that f satisfy the necessary conditions to presume the following operations):

- find the critical points; they are the points for which f_x and f_y vanish simultaneously;
- find the Hessian matrix of the function f;
- evaluate the Hessian matrix at the critical point(s);
- test the positivity of the Hessian matrix at the critical point(s). If the Hessian matrix is positively defined, then a relative minimum occurs at that critical point. If the Hessian matrix is negatively defined, then a relative maximum occurs.

2.2. **Maxima and minima for functions of two variables.** In this section we show how MAPLE V can be used to locate and test for local and global extrema for functions of two variables.

Problem 1. *Consider the function f defined by: $f(x, y) = x^3 + y^3 - 3xy$. Find the relative maxima and minima for function f.*

The function can be defined into a MAPLE V session by the following command:

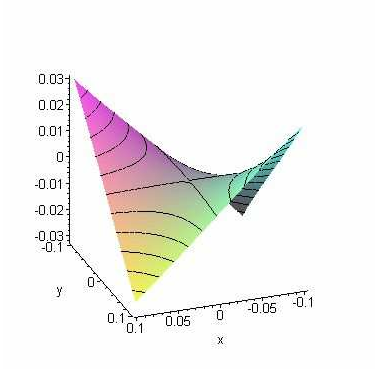

FIGURE 1

```
>with(student):with(linalg):
>f:=x^3+y^3-3*x*y;
```

In order to find the exact location of the critical points for the function f, we need to determine the first derivatives

```
>dfx:=diff(f,x);
 dfy:=diff(f,y);
```

The critical points are the points for which f_x and f_y vanish simultaneously

```
>sols:=solve({dfx=0,dfy=0},{x,y});
>s1:=sols[1];
 s2:=sols[2];
```

Thus, in this case there are two critical points: $(0,0)$ and $(1,1)$.

In order to characterize the critical points, (for every critical point) we evaluate the Hessian matrix at that point and then we test the positivity of this matrix

```
>H:=hessian(f,[x,y]);
>a:=subs(s1,H[1,1]);
 b:=subs(s1,H[1,2]);
 c:=subs(s1,H[2,2]);
>H1:=matrix(2,2,[a,b,b,c]);
>definite(H1,'func{positive}\_def');
 definite(H1,'func{negative}\_def');
>a:=subs(s2,H[1,1]);
 b:=subs(s2,H[1,2]);
 c:=subs(s2,H[2,2]);
>H2:=matrix(2,2,[a,b,b,c]);
>definite(H2,'func{positive}\_def');
 definite(H2,'func{negative}\_def');
```

In conclusion, $(0,0)$ is a saddle point and $(1,1)$ is a minimum point.

Geometric interpretation: The following commands illustrate how graphic visualization can be used to get an idea about extreme values

```
>plot3d(f(x,y),x=-3..3,y=-3..3,style=patch,orientation=[70,65],axes=FRAMED);
>plot3d(f(x,y),x=-0.1..0.1,y=-0.1..0.1,style=patchcontour,orientation=[70,65]);
>plot3d(f(x,y),x=0.9..1.1,y=0.9..1.1,style=patchcontour,orientation=[70,65]);
```

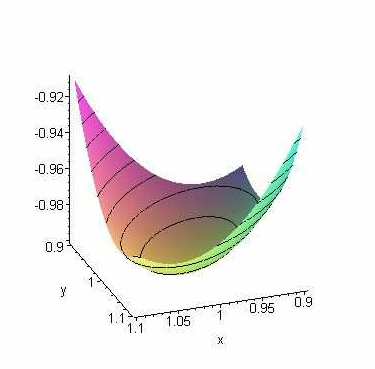

FIGURE 2

Indeed, these plots provided further evidence that the point $(0,0)$ is a saddle point (fig. 1) and at $(1,1)$ a relative minimum occurs (fig 2).

2.3. **Maxima and minima for functions of three variables.** The following example is the case of a function of three variables. We present here the MAPLE V commands necessary for the determination of the relative maxima or minima.

Problem 2. *Find the relative maxima or/and minima for the following function:*

$$f(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$$

```
>f:=x+(y^2)/(4*x)+(z^2)/y+2/z;
>with(student):with(linalg ):
>dfx:=diff(f,x);
 dfy:=diff(f,y);
 dfz:=diff(f,z);
>sols:=solve({dfx=0,dfy=0,dfz=0},{x,y,z});
>s1:=sols[1];
 s2:=sols[2];
>H:=hessian(f,[x,y,z]);
>a:=subs(s1,H[1,1]);b:=subs(s1,H[1,2]);c:=subs(s1,H[1,3]);
>d:=subs(s1,H[2,3]);e:=subs(s1,H[2,2]);f:=subs(s1,H[3,3]);
>H1:=matrix(3,3,[a,b,c,b,e,d,c,d,f]);
>definite(H1,'func{positive}\_def');
 definite(H1,'func{negative}\_def');
>a:=subs(s2,H[1,1]);b:=subs(s2,H[1,2]);c:=subs(s2,H[1,3]);
>d:=subs(s2,H[2,3]);e:=subs(s2,H[2,2]);f:=subs(s2,H[3,3]);
>H2:=matrix(3,3,[a,b,c,b,e,d,c,d,f]);
>definite(H2,'func{positive}\_def');
 definite(H2,'func{negative}\_def');
```

We deduce that $(\frac{1}{2},1,1)$ is a point of relative minimum and $(-\frac{1}{2},-1,-1)$ is a point of relative maximum.

3. CONSTRAINED OPTIMIZATION

3.1. **Mathematical concepts.** The optimization problems that have constraints are of the following type:

Problem 3. *Find the maximum or minimum of the function f(x,y,z) subject to the constraint g(x,y,z)=0.*

Here we consider the case of functions of three variables. We assume that f and g satisfy the necessary conditions for the following operations.

The algorithm is the following:

- form the "objective" function $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$;
- find the critical points of $f(x,y,z)$ which satisfy the constraint $g(x,y,z) = 0$. These are the points that make all the partial first derivatives equal to zero.

The following steps will be repeated for every critical point:

- evaluate d^2F at the critical point;
- evaluate dg at the critical point;
- evaluate dz from $dg = 0$ (at the critical point) in terms of dx and dy and introduce it in d^2F at the critical point;
- study d^2F for classifying the critical points. If d^2F is positively defined, then we have a minimum. If d^2F is negatively defined, then we have a maximum.

Remark 4. *If we have two constraintes $g_1(x,y,z) = 0$ and $g_2(x,y,z) = 0$, we must following the same procedure with the objective function: $F(x,y,z) = f(x,y,z) + \lambda_1 g_1(x,y,z) + \lambda_2 g_2(x,y,z)$.*

We find the critical points as previously and then we evaluate d^2F at the critical point(s). We solve the system (at the critical point) $\left\{ \begin{array}{l} dg_1 = 0 \\ dg_2 = 0 \end{array} \right\}$ and we evaluate dy and dz as a function of dx. We introduce these in d^2F and study the quadratic form d^2F in order to classify the critical points.

3.2. **A MAPLE V program for constrained optimization.** Consider the case of a function of three variables and with one constraint (the case of three variable functions and two constraints is similar; the program for this case can be find on the floppy disk which is attached to this paper).

Problem 5. *Find the extremum of the function $f(x,y,z) = xy + yz + xz$, subject to the constraint $xyz = 1$, $x > 0, y > 0, z > 0$.*

First, we define the functions f and g by using MAPLE V commands:

```
>with(student):with(linalg):
>f:=(x,y,z)-> x*y+y*z+x*z;
 g:=(x,y,z)-> x*y*z-1;$
```

Then we introduce the objective function F and solve the system for finding all the critical points.

```
>F:=(x,y,z)-> f(x,y,z)+lambda*g(x,y,z);
>dFx:=diff(F(x,y,z),x);
>dFy:=diff(F(x,y,z),y);
>dFz:=diff(F(x,y,z),z);
>sols:=solve({dFx=0,dFy=0,dFz=0,g(x,y,z)=0},{x,y,z,lambda});
>sols[1];
```

We calculate the first partial derivatives of the function g and all the second partial derivatives of the function F

```
>dgx:=diff(g(x,y,z),x);
 dgy:=diff(g(x,y,z),y);
 dgz:=diff(g(x,y,z),z);
>d2Fx:=diff(diff(F(x,y,z),x),x);
 d2Fy:=diff(diff(F(x,y,z),y),y);
>d2Fz:=diff(diff(F(x,y,z),z),z);
 d2Fxy:=diff(diff(F(x,y,z),x),y);
>d2Fxz:=diff(diff(F(x,y,z),x),z);
 d2Fyz:=diff(diff(F(x,y,z),y),z);
```

Evaluate the dz function of dx and dy from the equation $dg = o$ and introduce it in d^2F

```
>sol:=solve({dgx*dx+dgy*dy+dgz*dz=0},dz);
>d2F:=d2Fx*(dx)^2+d2Fy*(dy)^2+d2Fz*(dz)^2+2*d2Fxy*(dx)*(dy)+2*d2Fxz*(dx)*(dz)+2*d2Fyz*(dy)*(dz);
>subs(sol,d2F);
 p:=evala(subs(sol,d2F));
```

We study then the quadratic forme d^2F by using the MAPLE V command "definite"

```
>assign(sols[1]);p;a:=coeff(p,dx^2);b:=coeff(p,dy^2);c:=tcoeff(p);
>H:=matrix(2,2,[a,c/2,c/2,b]);definite(H,'func{positive}\_def');
>definite(H,'func{negative}\_def');$
>unassign('lambda');unassign('x');unassign('y'});unassign('z');
```

In conclusion (1,1,1) is a minimum point for function f subject to the constraint g.

4. Conclusions

Using computational techniques in every day teaching brings a lot of advantages; the student can better understand the significance of notions and of results of some exercises. Also, the results of some typical problems (analytical calculus) could be obtained too by those who do not master traditional techniques of calculus.

Some properties could be guessed by graphical analysis and next rigorously demonstrated (which is the case of the problems presented here - see *geometrical interpretations*).

The presented programms lead the student to change some instructions of the program for obtaining the result, proving the understanding of the correct algorithm.

For a better understanding of Mathematical Analysis and for learning the main methods of solving problems, MAPLE V can be very successfully used. This does not diminish the importance of theoretical knowledge, but fulfills it.

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Functional completeness in the simplest extensions of dual intuitionistic logic

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The Dual Intuitionistic Logic [1] (D.I.L.) is based on formulas constracted from symbols of variables p, q, r, s, possibly with supscripts, by means of three pairs of dual operators: & and $\vee$, $\neg$ and $\lrcorner$ (weak negation), $\supset$ and $-$ (substraction), and parentheses. Close conections of D.I.L. with the theory of distributive lattices were noticed in [2].

The formula F is said to be *expressible* in a logic L by means of the system Σ (of formulas) if F can be obtained, starting with variables and formulas of Σ, by means of weak rule of substitution or rule of replacement by equivalent elements in L. A system Σ is said to be *(functionally) complete* in a logic L if all formulas are expressible in L by means of Σ.

The conditions of completeness in chain extensions and in simplest non-chain extensions of D.I.L. were found out in [3, 4].

Let $m_0 = 1$ and let E_m mean $\{0, \tau_0, \tau_1, \dots, \tau_{m-2}\}$ (if m is finite), or $\{0, \tau_0, \tau_1, \tau_2, \dots\}$ (if m is ∞). Let E_m be a liniar ordered set $\tau_0 > \tau_1 > \tau_2 > \dots$.

Let us define the following operations on E_m: $p\&q = min(p,q)$, $p \vee q = max(p,q)$, $p \supset q = 1$ (if $p \leq q$) and $p \supset q = q$ (if $p > q$), $p - q = 0$ (if $p \geq q$) and $p - q = q$ (if $p < q$), $\neg p = p \supset 0$, $\lrcorner p = p - 1$.

Thus, we obtain the algebra

$$\mathfrak{A}_m =< E_m; \{\&, \vee, \supset, -, \neg, \lrcorner\} > .$$

The interpretation of formulas on this algebra leads us to some logic $L\mathfrak{A}_m$. It take place the relations

$$L\mathfrak{A}_2 \supseteq L\mathfrak{A}_3 \supseteq \dots \supseteq L\mathfrak{A}_i \supseteq \dots \supseteq D.I.L.$$

Theorem 1. *In order that a system (of formulas) Σ be functionally complete in the logic $L\mathfrak{A}_m$ ($m \geq 4$) it is necessary and sufficient that Σ be functionally complete in the logic $L\mathfrak{A}_4$*

Let consider the simplest non-chain algebra $Z_5 = \{0, \rho, \sigma, \omega, 1\}$ were the elements ρ and σ are incomparable, and $0 < \rho < \omega < 1$, $0 < \sigma < \omega$.

Theorem 2. *The system (of formulas) Σ is functionally complete in the logic $L\mathfrak{A}_5$ iff Σ is complete in $L\mathfrak{A}_3$, and for any of the 20 predicates indicated in [4] there exists in Σ a formula that does not preserve it on Z_5.*

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A variational principle for a hydromagnetic problem

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Abstract. A linear magnetic Bénard problem with tensorial electrical conductivity [1] governing the stability of the mechanical equilibrium of a viscous incompressible horizontal layer heated from below and acted upon a vertical magnetic field is shown to be equivalent to a variational problem. The given mathematical problem is formulated in Section 1 as a two-point problem for a system of ordinary differential equations. In Section 2 the adjoint of the matricial differential operator defining this problem is constructed while in Section 3 the functional defining the variational formulation equivalent to the given two-point problem is deduced.

1. Mathematical problem

Consider a homogeneous thermoellectrically conducting fluid situated in a horizontal layer S bounded by the planes $\pi_0 : z = 0$ and $\pi_1 : z = 1$, both stress-free, perfectly thermally and electricaly conductors. A constant vertical temperature is maintained in the presence of a uniform vertical magnetic field H_0. The dimensionless equations governing the perturbation u, h, θ, p of the thermodiffusive equilibrium m_0

$$m_0 \equiv \{\mathbf{U} = 0, \mathbf{H} = H_0\mathbf{k}, T = -\beta z + T_0, p_0 = p_0(z)\}$$

are

$$(1)\qquad \begin{cases} \frac{\partial}{\partial t}\mathbf{u} = -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + P_m\Delta\mathbf{u} + P_m M^2(\mathbf{H}_0 + \mathbf{h})\cdot\nabla\mathbf{h} + R\frac{P_m^2}{P_r}\theta\mathbf{k}, \\ \frac{\partial}{\partial t}\mathbf{h} = \nabla\times[\mathbf{u}\times(\mathbf{H}_0 + \mathbf{h})] + \Delta\mathbf{h} + \beta_H\nabla\times[(\mathbf{H}_0 + \mathbf{h})\times\nabla\times\mathbf{h}], \\ \frac{\partial}{\partial t}\theta = -\mathbf{u}\cdot\nabla\theta + \mathbf{u}\cdot\mathbf{k} + \frac{P_m}{P_r}\Delta\theta, \\ \nabla\cdot\mathbf{u} = 0, \\ \nabla\cdot\mathbf{h} = 0, \end{cases}$$

where $\mathbf{u}$ is the velocity field, $\mathbf{h}$ is the magnetic field, $\mathbf{k}$ is the upwards positive unit vector, θ is the temperature, p is the pression, the positive coefficients P_r, P_m, M^2 and R are the Prandtl, Prandtl magnetic, Hartmann and Rayleigh numbers respectively, β_H is the Hall coefficient occuring in the generalized Ohm's law.

With equations (1) we associate the boundary conditions corresponding to the specified type of fluid and flow. By performing the change of variables $z \to z - 0.5$ and assuming that the perturbations are normal modes, i.e.

$$(2)\qquad (w, h_3, j, \tau, \theta) = \{W(z), K(z), X(z), Z(z), \Theta(z)\}\exp[i(\alpha x + \beta y) + \sigma t],$$

where $w = k\cdot u$, $h_3 = k\cdot h$, $\tau = k\cdot\nabla\times u$, $j = k\cdot\nabla\times h$, the boundary value problem for the equations (1) becomes

$$(3)\qquad \begin{cases} (D^2 - a^2 - \sigma^2)K + DW - \beta_H DX = 0, \\ \left[P_m(D^2 - a^2) - \sigma\right]Z + P_m M^2 DX = 0, \\ (D^2 - a^2)\left[P_m(D^2 - a^2) - \sigma\right]W + P_m M^2 D(D^2 - a^2)K - R\frac{P_m^2}{P_r}a^2\Theta, \\ (D^2 - a^2 - \sigma)X + DZ + \beta_H D(D^2 - a^2)K = 0, \\ \left[\frac{P_m}{P_r}(D^2 - a^2) - \sigma\right]\Theta + W = 0, \end{cases}$$

$$(4)\qquad W = D^2W = K = DX = DZ = \Theta = 0,\ z = \pm 0.5,$$

where $a^2 = \alpha^2 + \beta^2$.

2. The adjoint problem

The problem (3) has the form $LU = 0$, where

$$L : \mathcal{D}(L) \to [C^\infty(-0.5; 0.5)]^5$$

is a linear matricial differential operator

$$\begin{pmatrix} D & D^2-a^2I-\sigma I & -\beta_H D & 0 & 0 \\ 0 & 0 & P_m M^2 D & P_m(D^2-a^2I)-\sigma I & 0 \\ (D^2-a^2)\left[P_m(D^2-a^2)-\sigma\right] & P_m M^2 D(D^2-a^2) & 0 & 0 & -R\frac{P_m^2}{P_r}a^2 I \\ 0 & \beta_H D(D^2-a^2) & D^2-a^2I-\sigma I & D & 0 \\ I & 0 & 0 & 0 & \frac{P_m}{P_r}(D^2-a^2I)-\sigma I \end{pmatrix},$$

I is the identity operator on $\mathcal{D}(L)$, $\mathcal{O}$ is the null operator,

$$\mathcal{D}(L) = \{U = (W, K, X, Z, \Theta), U \in [C^\infty[-0.5; 0.5]]^5 \text{ and satisfies(4)}\}.$$

The domain $\mathcal{D}(L) \subset \left[L^2(-0.5; 0.5)\right]^5$, L^2 is a Hilbert space of all functions f having the property $\int_{-0.5}^{0.5} f^2 dx < \infty$ (the integral is taken in the Lebesgue sense).

Remark. If a linear matricial nth order differential operator L, given by a n × n matrix (a_{ij}), where $a_{ij} = \sum\limits_{k=1}^{n} a_{ij}^k D^k$ ($D^k = \frac{\partial^k}{\partial x^k}$) and a_{ij}^k are constants, is selfadjoint the we must have $a_{ji} = \sum\limits_{k=1}^{n} (-1)^k a_{ij}^k D^k$. In our case, this condition is not fulfilled, hence the operator L is not selfadjoint. Let us now construct the adjoint operator of L. First remaind that the scalar product in $\left[L^2(-0.5; 0.5)\right]^5$ is, by definition,

$$(f, g) = \int_{-0.5}^{0.5} \sum_{i=1}^{5} f_i g_i dx, \ f, g \in \left[L^2(-0.5; 0.5)\right]^5.$$

Completing $\mathcal{D}(L)$ in the $\left[L^2[-0.5; 0.5]\right]^5$ norm we obtain a Hilbert space H, $\overline{\mathcal{D}(L)}^{\|\cdot\|_{L^2}} = H$.

The adjoint operator of L, denoted by $L^*, L^* : \mathcal{D}(L^*) \subset H \to H$ is defined by the relation $(L\mathbf{U}, \mathbf{U}^*) = (\mathbf{U}, L^*\mathbf{U}^*)$, where $\mathbf{U} \in \mathcal{D}(L)$ and $\mathbf{U}^* \in \mathcal{D}(L^*)$.

Here $\mathbf{U} = (W, K, X, Z, \Theta)$ and $\mathbf{U}^* = (W^*, K^*, X^*, Z^*, \Theta^*)$.

We have

$$\begin{aligned}(L\mathbf{U}, \mathbf{U}^*) = &\textstyle\int_{-0.5}^{0.5}(D^2 - a^2 - \sigma)KW^* + DWW^* - \beta_H DXW^* + \\ &+[P_m(D^2 - a^2) - \sigma]ZK^* + P_m M^2 DXK^* + \\ &+(D^2 - a^2)[P_m(D^2 - a^2) - \sigma]WX^* + P_m M^2 D(D^2 - a^2)KX^* - R\tfrac{P_m^2}{P_r}a^2\Theta X^* + \\ &+(D^2 - a^2 - \sigma)XZ^* + DZZ^* + \beta_H D(D^2 - a^2)KZ^* + \\ &+[\tfrac{P_m}{P_r}(D^2 - a^2) - \sigma]\Theta\Theta^* + W\Theta^*.\end{aligned}$$

Integrating by parts the expression $(L\mathbf{U}, \mathbf{u})$ and taking into account the boundary conditions (4) we obtain

$$\begin{aligned}(L\mathbf{U}, \mathbf{U}^*) = &[DW(P_m(D^2X^* - 2a^2X^*) - \sigma X^*) + D^3 W P_m X^* + \\ &+DK(W^* - P_m M^2 DX^* - \beta_H DZ^*) + D^2 K(P_m M^2 X^* + \beta_H Z^*) + \\ &+X(-\beta_H W^* + P_m M^2 K^* - DZ^*) + Z(-PmDK^* + Z^*) + \tfrac{P_m}{P_r} D\Theta\Theta^*]|_{-0.5}^{0.5} + \\ &+\textstyle\int_{-0.5}^{0.5}\{(D^2 - a^2)[P_m(D^2 - a^2) - \sigma]X^* - DW^* + \Theta^*\}W + \\ &+\{(D^2 - a^2 - \sigma)W^* - P_m M^2 D(D^2 - a^2)X^* - \beta_H D(D^2 - a^2 Z^*)\}K + \\ &+\{\beta_H DW^* - P_m M^2 DK^* + (D^2 - a^2 - \sigma)Z^*\}X + \\ &+\{[P_m(D^2 - a^2) - \sigma]K^* - DZ^*\}Z + \\ &+\{[\tfrac{P_m}{P_r}(D^2 - a^2) - \sigma]\Theta^* - R\tfrac{P_m^2}{P_r}a^2X^*\}\Theta.\end{aligned}$$

In order to have the equality $(L\mathbf{U},\mathbf{U}^*)=(\mathbf{U},L^*\mathbf{U}^*)$, taking into account the fact that the values of DW, D^3W, DK, D^2K, X, Z and $D\Theta$ are arbitrary at $z=\pm 0.5$, we obtain the boundary conditions

$$(5)\qquad \begin{array}{l} -\beta_H W^* + P_m M^2 K^* - DZ^* = W^* - P_m M^2 DX^* - \beta_H DZ^* = \\ = DK^* = X^* = D^2X^* = Z^* = \Theta^* = 0,\ z=\pm 0.5. \end{array}$$

Define $\mathcal{D}(L^*)=\{U^*\in[C^\infty(-0.5;0.5)]^5|\ U^*$ satisfies (5)$\}$.Then, from the equality $(L\mathbf{U},\mathbf{U}^*)=(\mathbf{U},L^*\mathbf{U}^*)$ it follows that the adjoint operator

$$L^*:\mathcal{D}(L^*)\to H$$

is defined by

$$\begin{pmatrix} -D & 0 & (D^2-a^2)\left[P_m(D^2-a^2)-\sigma\right] & 0 & I \\ D^2-a^2I-\sigma I & 0 & -P_mM^2D(D^2-a^2) & -\beta_H D(D^2-a^2) & 0 \\ \beta_H D & -P_mM^2D & 0 & D^2-a^2I-\sigma I & 0 \\ 0 & P_m(D^2-a^2I)-\sigma I & 0 & -D & 0 \\ 0 & 0 & -R\frac{P_m^2}{P_r}a^2I & 0 & \frac{P_m}{P_r}(D^2-a^2I)-\sigma I \end{pmatrix}.$$

In this way, the adjoint problem of (3), (4) is the two-point problem (6) for the system

$$(6)\qquad \begin{cases} (D^2-a^2)[P_m(D^2-a^2)-\sigma]X^* - DW^* + \Theta^* = 0, \\ (D^2-a^2-\sigma)W^* - P_mM^2D(D^2-a^2)X^* - \beta_H D(D^2-a^2Z^*) = 0, \\ \beta_H DW^* - P_mM^2DK^* + (D^2-a^2-\sigma)Z^* = 0, \\ \left[P_m(D^2-a^2)-\sigma\right]K^* - DZ^* = 0, \\ \left[\frac{P_m}{P_r}(D^2-a^2)-\sigma\right]\Theta^* - R\frac{P_m^2}{P_r}a^2x^* = 0, \end{cases}$$

In the following we assume that the principle of exchange of stabilities holds, i.e. $\sigma=0$. In this case, denoting $V_1^*=W^*-P_mM^2DX^*-\beta_HDZ^*$, equation $(6)_2$ and $(5)_2$ read $(D^2-a^2)V_1^*=0$ and $V_1^*(\pm 0.5)=0$ respectively implying $V_1^*\equiv 0$ in $[-0.5,0.5]$. This is why the boundary condition $(5)_2$ is considered no longer.

3. Variational principle for the problem (3), (4)

By a variational principle we mean a theorem which establishes the equiva-lence between the set of solutions of a boundary value problem and the set of stationary points of a corresponding functional [2], [3].
Denote by $J(U,u)$ the functional $J:\mathcal{D}(L)\times\mathcal{D}(L^*)\to\mathbb{R}$,

$$(7)\qquad J(U,U^*)=\int_a^b L_1U(x)L_2U^*(x)dx$$

and assume that

$$(8)\qquad (LU,U^*)=\int_a^b L_1U(x)L_2U^*(x)dx,$$

$L_1:\mathcal{D}(L)\to H, L_2:\mathcal{D}(L^*)\to H$ and

$$(9)\qquad \int_a^b L_1U(x)L_2U^*(x)dx=\int_a^b U(x)L^*U^*(x)dx=(U,L^*U^*).$$

Let us apply to J the Lagrange variation to get

$$\delta J(U,U^*)=\int_a^b[\delta(L_1U)L_2U^*+L_1U\delta(L_2U^*)]dz.$$

By this assumption we have $\int_a^b \delta(L_1U)L_2U^* = \int_a^b \delta(U)L^*U^*dx$ and

$$\int_a^b (L_1U)\delta(L_2U^*)) = \int_a^b (LU)\delta U^*dx.$$

Consequently

$$\delta J(U,U^*) = \int_a^b LU\delta U^* + \int_a^b L^*U^*\delta U. \tag{10}$$

In this way, $\delta J = 0$ iff $LU = 0$ and $L^*U^* = 0$. This variational principle is conditioned by the possibility to obtain (8) and (9), by using part integration and the boundary conditions occurring in the definition of $\mathcal{D}(L)$ and $\mathcal{D}(L^*)$.
In our case, by appropriate by part integrations and taking into account the conditions (4) and $(5)_1$, $(5)_{3-7}$, we obtain

$$(LU,U^*) = J(U,U^*) = (U,L^*U^*),$$

where $a = -0.5, b = 0.5$, L and L^* are the operators defined in Section 2. Consequently we proved.
Theorem. $LU = 0, L^*u = 0$ iff $\delta J(U,u) = 0, \forall U \in \mathcal{D}(L), \forall U^* \in \mathcal{D}(L^*)$.

4. Fourier series solution

In order to solve the variational problem for $\delta J = 0$ we can use the direct Fourier techniques so that the expansion functions satisfy or not all the boundary conditions of the problem, i.e. the Chandrasekar method or the Budianski-DiPrima method respectively.

This method is used especially when the direct and the adjoint equations are the same and, consequently, the corresponding functional J is symmetric and, thus, L_1 and L_2 have the same expression(irrespective the relationship between their domain of definition). The expressions for the Fourier coefficients of the derivatives of the unknown functions become more complicated as the order is increased. So, the best variational principles is that one whose corresponding functional contains the lowest order derivatives possible. Therefore, the best variational principle is one in which the functional is symmetric. For nonsymmetric functionals (as in our case)the best varia-tional principle in that one involving the lowest order derivatives; several such principle can exist. Formally the functional J is obtained by performing by part integrations until the relation

$$(LU,U^*) = J(U,U^*) \tag{11}$$

is obtained. This possibility strongly depends on the boundary conditions, assuming that the sum of free terms (followed as a result of the by part integration and taken at $z = \pm 0.5$) vanishes. If (11) cannot be obtained, i.e. some terms computed at $z = \pm 0.5$ are left, then the chosen functional is not appropriate. In fact, in the second case an additional integration was performed enabling us to get more terms, free terms and thus their sum vanish in view of (4) and (5).
Remark. In applying the Budianski-DiPrima method [1], [4], the functions V_1^* defined in Section 2 and $V_2^* = -\beta_H W^* + P_m M^2 K^* - DZ^*$ occurring in the boundary conditions (5) were revealed also in [1] and [4] in the system in the Fourier coefficients. We think that a change of variable involving V_1^* and V_2^* could simplify the computation. This idea will be exploited elsewhere.

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Nonhyperbolic singularities in a system of two coupled oscillators

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Abstract. A system of two coupled identical oscillators, each of them being an advertising model, is considered. It possesses 4 variables and it depends on 3 parameters. We found the existence of a single symmetric equilibrium point, which is the origin, and of 4 nonsymmetric equilibria. Analyzing the eigenvalues of the liniarized system around each equilibrium point, the locus in the parameters space corresponding to nonhyperbolic singularities is determined.

1. Single advertising model

Consider the advertising model consisting in the Cauchy problem for the system of two nonlinear ordinary differential equations (ODEs) [4], [5]

$$\left\{ \begin{array}{l} \dot{x} = k - \gamma xy + \beta y, \\ \dot{y} = \gamma xy - \delta y. \end{array} \right. \tag{1}$$

Here $x(\tau)$ is the number of potential buyers at the moment τ, $y(\tau)$ is the number of the brand users at the moment τ, $\gamma(\tau) = \alpha y(\tau)$ is the contact rate with the advertising at time τ, β is the switching rate to rival brand and $\delta = \beta + \varepsilon$, where ε is the migration or mortality.

Using the transformations

$$u = \frac{\alpha k}{\delta \varepsilon} x - 1, v = \frac{\varepsilon}{k} y - 1, a = \frac{\alpha k^2}{\delta \varepsilon^2}, b = 2 - \frac{\beta}{\delta}, t = \delta \tau \tag{2}$$

system (1) becomes

$$\left\{ \begin{array}{l} \dot{u} = -a\left(u + bv + 2uv + v^2 + uv^2\right), \\ \dot{v} = u + v + 2uv + v^2 + uv^2, \end{array} \right. \tag{3}$$

where the dot stands for the differentiation with respect to the new time t. As $\delta > \beta$, we have $b > 1$, so the only case of interest from the applications point of view is the case

$$a > 0, \quad b > 1. \tag{4}$$

As the equilibrium points of (3) satisfy $\dot{u} = \dot{v} = 0$, under the assumptions (4) the only equilibrium point is $(0,0)$. The Jacobi matrix of the linearized system around $(0,0)$ is $A = \begin{pmatrix} -a & -ab \\ 1 & 1 \end{pmatrix}$ and the corresponding characteristic equation reads

$$\lambda^2 + \lambda(a-1) + a(b-1) = 0. \tag{5}$$

Its discriminant is $\Delta = a^2 - 4ab + 2a + 1$. The curve $\Gamma : \Delta = 0$ of the (a,b) plane, is dividing the domain defined by (4) into regions where the eigenvalues are real or not. Analyzing the sign of trA and $\det A$, we obtain the nature of the eigenvalues and the sign of their real parts, and thus the topological type of the equilibrium, as follows (fig. 1):

- region I: $trA > 0$, $\det A > 0, \Delta > 0$, so $\lambda_{1,2} \in R_+$ and $(0,0)$ is a repulsive node;
- region II: $trA > 0$, $\det A > 0, \Delta < 0$, so $\lambda_{1,2} \notin R$, $\mathrm{Re}\lambda_{1,2} > 0$ and $(0,0)$ is a repulsive focus;
- region III: $trA < 0$, $\det A > 0, \Delta < 0$, so $\lambda_{1,2} \notin R$, $\mathrm{Re}\lambda_{1,2} < 0$ and $(0,0)$ is an attractive focus;
- region IV: $trA < 0$, $\det A > 0, \Delta > 0$, so $\lambda_{1,2} \in R_-$ and $(0,0)$ is an attractive node.
- for $(a,b) \in H_0 = \{(a,b), a = 1, b > 1\}$, we get $trA = 0$, $\det A > 0, \Delta < 0$, so $\lambda_{1,2} = \pm i\sqrt{b-1}$, and $(0,0)$ is nonhyperbolic of Hopf type.

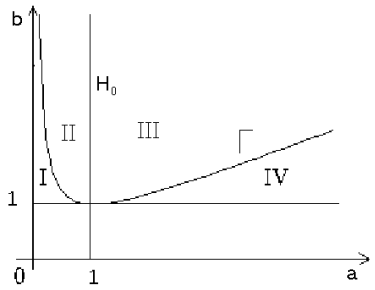

FIGURE 1. Regions of the parameter domain where the nature and sign of the real parts of the eigenvalues are preserved.

As it is shown in [4], [5], when crossing the line H_0 a Hopf bifurcation takes place and an attractive limit cycle surrounds the repulsor in regions II and I.

2. Coupled advertising models

Our study is designed to emulate two advertising models linked via the flow of potential buyers. Thus, we choose the form of this coupling to be a constant multiplied by the difference of potential buyers of the two brands. This leads to a system of four coupled, nonlinear ODEs

$$\begin{cases} \dot{u}_1 = -a_1\left(u_1 + b_1 v_1 + 2u_1 v_1 + v_1^2 + u_1 v_1^2\right) + c_{12}\left(u_1 - u_2\right), \\ \dot{v}_1 = u_1 + v_1 + 2u_1 v_1 + v_1^2 + u_1 v_1^2, \\ \dot{u}_2 = -a_2\left(u_2 + b_2 v_2 + 2u_2 v_2 + v_2^2 + u_2 v_2^2\right) + c_{21}\left(u_2 - u_1\right), \\ \dot{v}_2 = u_2 + v_2 + 2u_2 v_2 + v_2^2 + u_2 v_2^2. \end{cases} \tag{6}$$

We focus on the case when the parameter values are such that both models exhibit the same qualitative behavior. Thus, we assume that the two models are identical (i.e. $a_1 = a_2 = a, b_1 = b_2 = b$) and the coupling is symmetric (i.e. $c_{12} = c_{21} = c$). In this case, (6) reads

$$\begin{cases} \dot{u}_1 = -a\left(u_1 + b v_1 + 2u_1 v_1 + v_1^2 + u_1 v_1^2\right) + c\left(u_1 - u_2\right), \\ \dot{v}_1 = u_1 + v_1 + 2u_1 v_1 + v_1^2 + u_1 v_1^2, \\ \dot{u}_2 = -a\left(u_2 + b v_2 + 2u_2 v_2 + v_2^2 + u_2 v_2^2\right) + c\left(u_2 - u_1\right), \\ \dot{v}_2 = u_2 + v_2 + 2u_2 v_2 + v_2^2 + u_2 v_2^2, \end{cases} \tag{7}$$

A consequence of the above assumptions is the invariance of equations under the transformation $(u_1, v_1, u_2, v_2) \leftrightarrow (u_2, v_2, u_1, v_1)$. This symmetry can also be seen in the existence of an invariant space for equation (7), namely $S = \{(u_1, v_1, u_2, v_2), u_1 = u_2, v_1 = v_2\}$. Equilibria which lie in S will be reffered as symmetric and those which are not in S as nonsymmetric solutions. It is sufficient to consider $c \geq 0$. Taking into account (4), the parameter domain is

$$D = \{(a, b, c), a(b-1) > 0, c \geq 0\}. \tag{8}$$

We begin our study by determining the equilibrium points of (7). Under the assumption (8), there exists a unique symmetric equilibrium point $e_0 = (0, 0, 0, 0)$. Nonsymmetric equilibrium points are $e_1 = \left(-\frac{1}{2}, 1, -\frac{1}{2} - \frac{a(b-1)}{c}, -1\right)$, $e_2 = \left(-\frac{1}{2} - \frac{a(b-1)}{c}, -1, -\frac{1}{2}, 1\right)$, $e_{3,4} = \left(p, -\frac{p}{p+1}, -\frac{p}{2p+1}, \frac{p}{p+1}\right)$, where p satisfies the equation

$$c + \frac{c}{2p+1} + \frac{a\,(b-1)}{p+1} = 0, \tag{9}$$

or, equivalently

$$2cp^2 + (2p+1)\left[2c + a\,(b-1)\right] = 0. \tag{10}$$

Note that nonsymmetric equilibria exist only for $c \neq 0$.

As the discriminant of equation (10) is $a(b-1)[2c+a(b-1)]$ and it is always positive in D, it follows that system (7) has four nonsymmetric equilibria for any $(a,b,c) \in D$, $c \neq 0$. The Jacobi matrix of system (7) can be written in the form

$$J = \begin{pmatrix} -ag(v_1)+c & -a[b+h(u_1,v_1)-2] & -c & 0 \\ g(v_1) & h(u_1,v_1)-1 & 0 & 0 \\ -c & 0 & -ag(v_2)+c & -a[b+h(u_2,v_2)-2] \\ 0 & 0 & g(v_2) & h(u_2,v_2)-1 \end{pmatrix},$$

with $g(v) = (1+v)^2$ and $h(u,v) = 2(u+1)(v+1)$.

3. The symmetric equilibrium point e_0

The Jacobi matrix of the linearization of (7) around e_0 is

$$J(e_0) = \begin{pmatrix} -a+c & -ab & -c & 0 \\ 1 & 1 & 0 & 0 \\ -c & 0 & -a+c & -ab \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and the corresponding characteristic equation reads

$$\left[\lambda^2 + \lambda(a-1) + a(b-1)\right]\left[\lambda^2 + \lambda(a-2c-1) + a(b-1) + 2c\right] = 0.$$

Thus, the four eigenvalues of $J(e_0)$ come in two pairs: $\lambda_{1,2}$ satisfy (5), so they are eigenvalues for A in the case of a single model, while $\lambda_{3,4}$ satisfy

$$\lambda^2 + \lambda(a-2c-1) + a(b-1) + 2c = 0. \tag{11}$$

Note that for $c = 0$ equation (11) becomes (5). In this case, $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$ and the corresponding eigenvectors are lying in the invariant subspace S.

The equilibrium e_0 is nonhyperbolic if one of the eigenvalues λ_i, $i = \overline{1,4}$, has a zero real part. Since in D we have

$$a(b-1) > 0, \quad 2c + a(b-1) > 0,$$

it follows that equations (5), (11) have no zero solution. Hence, the equilibrium e_0 may be nonhyperbolic only if (5) or (11) have a pair of pure imaginary solution, that is in one of the following situations:

(1) $c = 0$, $a = 1$, $b > 1$; for these values e_0 has two pairs of pure imaginary eigenvalues $\lambda_1 = \lambda_3 = -\lambda_2 = -\lambda_4 = i\sqrt{b-1}$. Taking into account the form of $J(e_0)$, it follows that the parameters situated on the curve H_0 (fig.2a) in the $c = 0$ plane may correspond to D_2- symmetric Hopf bifurcation [2], [3].

(2) $c \neq 0$, $a = 1$, $b > 1$; for these values $\lambda_1 = -\lambda_2 = i\sqrt{b-1}$, $\mathrm{Re}\lambda_3 \cdot \mathrm{Re}\lambda_4 \neq 0$. It follows that the parameters (a,b) situated on the curve $H_0 : a = 1$, $b > 1$, in the plane $c = const.$ (fig. 2b), may correspond to a Hopf bifurcation.

(3) $c \neq 0$, $a = 2c+1$, $a(b-1) + 2c > 0$; for these parameter values λ_3, λ_4 are pure imaginary, $\mathrm{Re}\lambda_1 \cdot \mathrm{Re}\lambda_2 \neq 0$. Thus, for the parameters (a,b) situated on the curve $H_1 : a = 2c+1$, (fig. 2b) in a $c = const.$ plane may correspond to a Hopf bifurcation.

4. The nonsymmetric equilibrium points $e_{1,2}$

The Jabobi matrix of the linearization of (7) around the equilibrium e_1 reads

$$J(e_1) = \begin{pmatrix} c-4a & -ab & -c & 0 \\ 4 & 1 & 0 & 0 \\ -c & 0 & c & -a(b-2) \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and the corresponding characteristic equation is

$$(\lambda+1)\left[\lambda^3 - \lambda^2(2c-4a+1) + 2\lambda(c-2ac+2ab-2a) - 4ac(b-1)\right] = 0. \tag{12}$$

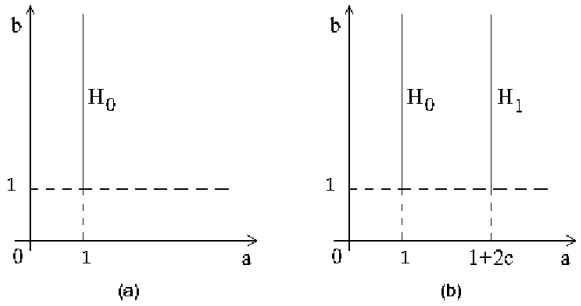

FIGURE 2. (a) The curve H_0 of possible symmetric Hopf bifurcations values, for $c = 0$; (b) The curves H_0, H_1 of parameters for which e_0 is nonhyperbolic, for $c > 0$.

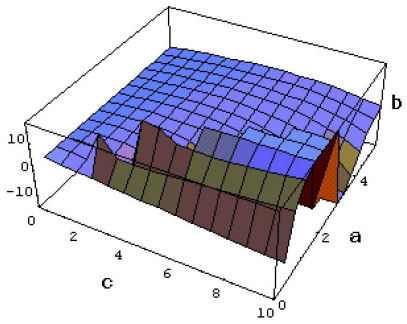

FIGURE 3. The surface H_2 of parameter values for which $e_{1,2}$ are nonhyperbolic.

It follows that $\lambda_1 = -1$ and $\lambda_2, \lambda_3, \lambda_4$ satisfy

$$\lambda^3 - \lambda^2 (2c - 4a + 1) + 2\lambda (c - 2ac + 2ab - 2a) - 4ac (b - 1) = 0. \tag{13}$$

Thus, $\lambda_2\lambda_3\lambda_4 = 0$ only if $ac(b-1) = 0$. Since by our assumption (8) $ac(b-1) \neq 0$, it follows that (12) has no zero solutions. Hence, the equilibrium e_1 may be nonhyperbolic only if (13) has a pair of pure imaginary solutions, say λ_2, λ_3. In such a case, from $\lambda_2 + \lambda_3 + \lambda_4 = 2c - 4a + 1$ we obtain

$$\lambda_4 = 2c - 4a + 1$$

and from $\lambda_2\lambda_3\lambda_4 = 4ac(b-1)$, it follows

$$\lambda_2\lambda_3 = \frac{4ac(b-1)}{2c - 4a + 1} > 0.$$

Taking into account that $\lambda_4(\lambda_2 + \lambda_3) + \lambda_2\lambda_3 = 2(c - 2ac + 2ab - 2a)$, we finally obtain the conditions for which e_1 has a pair of pure imaginary eigenvalues:

$$\begin{cases} \frac{4ac(b-1)}{2c-4a+1} = 2(c - 2ac + 2ab - 2a) \\ 2c - 4a + 1 > 0 \end{cases} \tag{14}$$

This takes place for parameters situated on the surface (fig. 3)

$$H_2 : b = 1 + \frac{c\,(2c - 4a + 1)\,(2a - 1)}{2a\,(c - 4a + 1)}, \quad 0 < a < \frac{2c + 1}{4}.$$

Sections with planes $c = const.$ in the surface H_2 are represented in fig. 4 and are denoted by H_2, too. Of course, only the branches of H_2 with $b > 1$ must be considered. As the Jacobi matrix corresponding

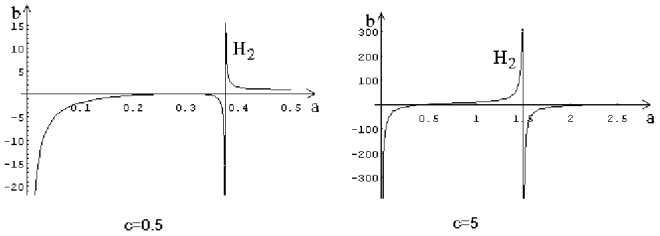

FIGURE 4. The curve H_2 of parameter values for which $e_{1,2}$ are nonhyperbolic.

to e_2 is

$$J(e_2) = \begin{pmatrix} c & -a(b-2) & -c & 0 \\ 0 & -1 & 0 & 0 \\ -c & 0 & c-4a & -ab \\ 0 & 0 & 4 & 1 \end{pmatrix},$$

the characteristic equation is also (12). Thus, if (14) holds, both e_1 and e_2 are Hopf singularities (degenerated or not).

5. The nonsymmetric equilibrium points $e_{3,4}$

For the equilibrium points $e_{3,4}$ we have: $1+u_1 = 1+p$, $1+v_1 = \frac{1}{p+1}$, $1+u_2 = \frac{p+1}{2p+1}$, $1+v_2 = \frac{2p+1}{p+1}$. Taking into account (9) and (10), the Jacobi matrix J corresponding to $e_{3,4}$ has the form

$$J(e_{3,4}) = \begin{pmatrix} \frac{2c}{(b-1)(2p+1)} + c & -ab & -c & 0 \\ -\frac{2c}{a(b-1)(2p+1)} & 1 & 0 & 0 \\ -c & 0 & \frac{2c(2p+1)}{b-1} + c & -ab \\ 0 & 0 & -\frac{2c(2p+1)}{a(b-1)} & 1 \end{pmatrix}.$$

The corresponding characteristic equation is written as

$$\lambda^4 - \Delta_1\lambda^3 + \Delta_2\lambda^2 - \Delta_3\lambda + \Delta_4 = 0, \tag{15}$$

where:

$$\begin{aligned} \Delta_1 &= -\frac{2}{b-1}\left(-b-2a+3c-cb+2ab+1\right); \\ \Delta_2 &= -\frac{1}{(b-1)^2}(1-8a-2b+12c-20cb+8abc+8c^2+b^2+4ab^3 \\ &\quad -4ac-16ab^2+20ab-4c^2b-4ab^2c+8cb^2); \\ \Delta_3 &= -\frac{2}{(b-1)}\left(-3c+4ac+3cb-8c^2+2a-6abc+2c^2b+2ab^2-4ab+2ab^2c\right); \\ \Delta_4 &= 4c\left[a\left(b-1\right)+2c\right]. \end{aligned}$$

Since $\Delta_4 > 0$ for $(a,b,c) \in D$, $c \neq 0$, it follows $\lambda_i \neq 0$, $i = \overline{1,4}$. Therefore, the equilibria $e_{3,4}$ may be nonhyperbolic only if (15) has a pair of pure imaginary solutions.

The conditions for the existence of a single pair of pure imaginary eigenvalues are

$$\Delta_1 \neq 0, \frac{\Delta_3}{\Delta_1} > 0, \frac{\Delta_3}{\Delta_1} + \Delta_4\frac{\Delta_1}{\Delta_3} = \Delta_2 \tag{16}$$

or

$$\Delta_1 = 0, \Delta_3 = 0, \Delta_4 < 0. \tag{17}$$

Conditions (17) are not fulfilled for parameters in D.

Equation (15) has two pairs of pure imaginary solutions if

$$\Delta_1 = 0, \Delta_3 = 0, \Delta_2 > 0, \Delta_4 > 0. \tag{18}$$

Denote by H_3 the set of parameters $(a, b, c) \in D$ satisfying

$$(\Delta_3)^2 + (\Delta_1)^2 \Delta_4 = \Delta_1 \Delta_2 \Delta_3.$$

Hence the parameter values satisfying conditions (16), or (18) are situated on the surface H_3. For a fixed c, conditions (16) are fulfilled for (a, b) situated on a curve H_3, while (18) are satisfied for at most three points of the curve H_3.

This paper is a first step in the study of coupled advertising models.

We have determined the equilibrium points for two coupled identical advertising models and we have identified the parameter values for which this system possesses nonhyperbolic equilibria.

Our study will continue in two directions. First we shall investigate the bifurcations and the dynamic behavior determined by the presence of nonhyperbolic equilibria. Second, we intend to perform a similar study for two coupled nonidentical advertising models.

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Large computer modeling of the biogas generation process

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During the latest decade the volume of solid waste generation has essentially increased. This fact is connected with the abrupt consumption rise, which is observed in industrial countries. At present the flow mass of solid waste, which annually is forthcoming into biosphere, has reached almost geological scale and amounts approximately 400 million tons per year [1]. All growing quantity of waste products has a sharp effect on global geochemical cycles of a lot of elements containing organic chemistry, in particular organic carbon. So, the weight of this element, forthcoming into an environment with waste products, makes approximately 85 million tones per one year while the general natural inflow of carbon into a soil cover of a planet makes only 41.4 million tons per one year.

One of the basic ways of solid waste removal all over the world remains its burial in the near-surface geological environment. In these conditions waste products are exposed to intensive biochemical decomposition which causes in particular landfill gas generation. The issues of biogas forthcoming into the natural environment form negative effects of both local and global character.

The quantity of the evolved biogas, and also the speed of the gas generation process are determined by the conditions of the environment which have been usual in concrete dump body. The humidity, temperature, composition of organic fractions concern the number of parameters rendering essential influence on decomposition of organic chemistry. Their complex influence is reflected in the following kinetics equation of the first order for the gas generation reaction [1]

$$Q = Mqe^{-kt} \tag{1}$$

- Q – quantity of biogas (m^3), formed in time t (years);
- M – weight of waste products (t);
- q – specific gas potential (m^3/t);
- k – a constant of speed of gas generation reaction (1/year).

In practice, various modifications of the formula (1) are applied for the forecasting of gas generation. Their basic distinction is reduced to the quantity of fractions of the solid waste organic substance included into consideration. These fractions essentially differ in their physical-chemical properties and terms of biochemical disintegration. So, "fast" fractions decay during 2-4 years, and the slower ones - during decades.

In work [2] the mathematical equation for an estimation of issue of biogas in England, for the period from 1970 till 2000 is given. It is based on the model of National Physical Laboratory (NPL). Given model allows carrying out an estimation of issue of biogas depending on the row of the determining factors.

In the present paper the mathematical model and the computer program of process of decomposition of organic chemistry and of biogas formation is yielded. From the mathematical point of view it represents a system of partial differential equations. The given model allows us to take into account the key parameters influencing the gas capacity and to carry out the forecasting and monitoring of mentioned processes.

Various research organizations in many countries of the world are actively engaged in the search of alternative sources of energy. There are many nonconventional energy sources: solar, wind, the heat of underground sources, the tidal energy in the seas and oceans etc. An energy source of anthropogenous character is of great importance now. It is the biogas received from organic components of household waste. By various expert estimations the organic components make $\sim 70\%$ of all waste products. Thus, usual dumps of waste are the powerful biological reactors capable to make the valuable energy carrier – biogas – within decades.

Practically in all industrially advanced countries, and also in many developing ones there are national programs of getting the biogas as a result of decomposition of organic substances and its further use.

In the paper we make an attempt to describe the influence of temperature, humidity, density and other parameters on the process of decomposition of organic chemistry by methods of mathematical modeling.

In many countries of the world the interest to biogas usage has increased in connection with the increase of prices for energy carriers. The biogas is formed of organic constituents of waste products at their decomposition by microorganisms. The process of decomposition proceeds in two stages: in the presence of oxygen the process is aerobic, and after an exhaustion of oxygen in deeper layers (oxygen-free) the anaerobic stage begins. The biogas evolved by microorganisms contains $\sim 60\%$ of methane and may be used on a place as low-calorie fuel, or may be processed into high-calorific gas and used for needs of gas supply.

Positive experience on biogas creation and use is received in many countries of the world. So, in USA the general number of municipal dumps has increased from 19.000 in 1985 up to 23.000 in 1995, thus the share of dumps with the tonnage of waste more than 1 million tons has increased from 9% up to 23%. This is essential from the point of view of economy of biogas usage. The computations of the economic efficiency of biogas usage carried out in 1982 for dumps with various volumes of stored waste, show that for volumes more than 1 million tons the use of biogas becomes profitable. In Moldova, taking into account the lower wages of workers and the world prices for gas, it is economically expedient to maintain dumps with the smaller contents of waste products.

The mathematical model constructed on the basis of the factors influencing on the decomposition of organic chemistry and on manufacture of biogas allows us to carry out necessary calculations and to predict behaviour of the modeled system. The values for initial and boundary conditions of basic modeled parameters are obtained experimentally at Academy of the municipal Services of a name of K.D.Pamfilov. As a result of the carried out experimental researches and the subsequent laboratory processing the functional dependencies of changes of humidity and ash content along depth, the contents of organic chemistry, density, and also values of heat conductivity and a thermal capacity of samples in a damp and dry condition are obtained.

As the researches show, the biogas spreads in the thickness of the stored waste products mainly in the horizontal direction (at presence of insulating layers). The exit (or issue) of biogas from a surface into atmosphere is determined by a number of factors. If the top layer of a dump is insufficiently dense then the evolved gas is mixed up with atmospheric air. At good condensation or at presence of a tight covering of a surface the biogas is concentrated and evolved basically on slopes and on the limited sites of a surface.

Two kinds of degassing of dumps are applied to practical use of biogas:

1. passive degassing which is carried out due to the own pressure available in thickness of a dump;
2. active degassing which is carried out with the help of special devices for extraction of gas.

Passive degassing is seldom applied to get biogas because of its low efficiency. The increased requirements for prevention of uncontrollable issue of gas into an atmosphere are made to the means of active degassing.

The basic results received at modeling of conditions, arising on solid waste landfills are presented below. One of the advantages of mathematical modeling methods consists in the fact that we may define the influence of one parameter, having fixed the others. At the given stage of researches the influence of the following factors is investigated:

- contents of organic chemistry in waste products;
- quantity of humidity;
- influence of seasonal difference of temperatures.

The investigations were carried out by using accessible experimental data received basically from the landfill Brateevo, Moscow. For forecasting the speed of decomposition of organic components at other landfills it is necessary to have the appropriate experimental material.

Besides, it is necessary to model the natural conditions arising during the waste products storage. In addition, the seasonal changes of temperature are to be considered. For simplicity the harmonious

fluctuations in limits from T_{min} up to T_{max}, i.e. the minimal were used and maximal values of temperatures in the summer and in the winter time. The constant ambient temperature $12°C$ exists at the lower bound.

The distribution of temperature, which arises in a real ground without sources of heat, was accepted in the model as the initial one. Additional experimental data can be used as starting conditions for the problem of heat conductivity. We note that the only effect on the solution concernes the terms of establishment of a temperature mode and the initial speed of ageing of a dump.

In order to study the temperature fields at the given stage of researches the modeling variant which is taking into account the lamination of environment, the action of heat sources and the dependence of temperature conductivity factor of the environment from humidity was selected.

The purpose of the present paper is the computation of the distribution of temperature along the depth of the researched layer. For the description of the process of heat distribution the equation of heat conductivity

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(\lambda(T)\frac{\partial T}{\partial z}\right) + Q(z). \tag{2}$$

is used.

This equation is of parabolic type and if dependencies $\lambda(T)$ and $Q(z)$ are known, then it is possible to solve the equation (2) numerically. Parameters $\lambda(T)$ and $Q(z)$ are the functional dependencies on temperature, humidity and other parameters of environment. They may be obtained by numerical approximation of experimental data.

Initial and boundary conditions. The initial conditions of the problem consist of the given distribution of temperature at an initial period of time $t = 0$

$$T(z, 0) = T_0(z) \tag{3}$$

and through distribution of humidity:

$$W(z, 0) = W_0(z). \tag{4}$$

In practice, naturally, it is difficult to define experimentally the functions $T_0(z)$ or $W_0(z)$ way at the moment the investigated layer has been initiated, or at any other moment of time. However, the requirements of the mathematical methods, are not too strict from the given point of view. As we shall see, after several iterations the solution is got with required accuracy even if functions do not correspond to conditions of uniformity of environment.

The situation is complicate a little bit in the case of lamination of environment. The iterative process converges in this case too. However, to get necessary accuracy more iterations one needs. As boundary conditions the conditions of 1-st, 2-nd or 3-rd sort may be used. From the physical point of view the boundary conditions of 2-nd sort are more correct as they describe behaviour of a stream of heat on the boundaries. This is supposed to be proportional to the difference of temperatures

$$\frac{\partial T}{\partial z} = h_1(T - T_c), z = 0 \tag{5}$$

$$\frac{\partial T}{\partial z} = h_2(T - T_z), z = l, \tag{6}$$

where T_c and T_z are the ambient temperatures of this layer.

The model (2) - (6) represents the initial-boundary value problem for a partial differential equation. Its solution may be determined by a method of finite differences. The decision is searched in area $R : \{0 \le z \le l, t > 0\}$ with the initial and boundary conditions (4) - (6).

The part of the results, obtained by computer experiments carried out with the constructed model, is presented in fig. 1.

Fig.1.a shows the results of the computations which match the changes of temperature field depending on time (in years). The contents of organic chemistry in the waste is equal to 800 kg/m^3. The temperature curve 1 corresponds to the depth of 5m, the curve 2 – to 10m, the curve 3 – to 15m.

Fig.1.b shows the temperature dependence on depth. The contents of organic chemistry in the waste is equal to 600 kg/m^3. The curves 1,2,3,4,5 are calculated for fall in 1, 2, 3, 10 and 20 years correspondingly after modeling the starting.

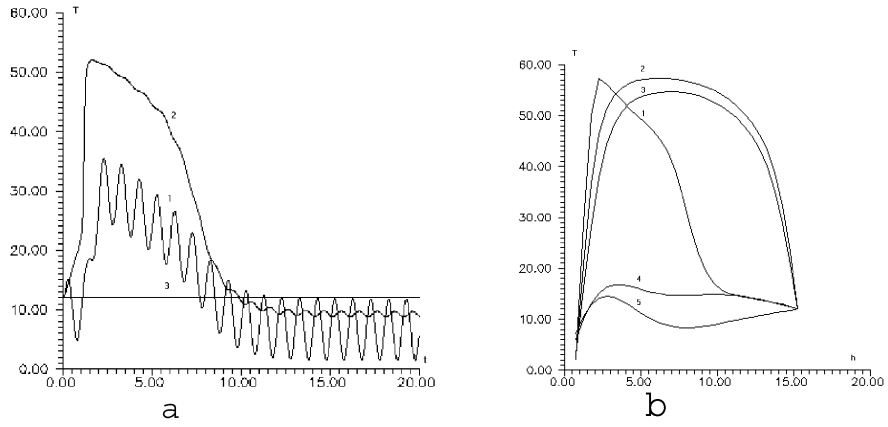

FIGURE 1

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A q - analogue of Lupaş operators

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Abstract. In order to approximate the function $f:[0,\infty)\to\mathbb{R}$, with $|f(x)|\leq A\cdot 2^x$ for $x>0$, A. Lupaş introduced in [8] the approximation operators

$$(\mathcal{L}_n f)(x)=2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}f\left(\frac{k}{n}\right),\quad x\geq 0,$$

where $(z)_0=1$, $(z)_n=z(z+1)\dots(z+k-1)$ for $k\geq 1$.
Our aim is to find a q - analogue of these operator.

1

For $q\in\mathbb{C}\backslash\{1\}$, let us denote by $[\mathbf{n}]_\mathbf{q}=\dfrac{q^n-1}{q-1}$, and for $n\in\mathbb{N}$

$$[\mathbf{n}]_\mathbf{q}!=\begin{cases}1 & \text{if} \quad n=0\\ [1]_q[2]_q\dots[n]_q & \text{if} \quad n=1,2,\dots,\end{cases},$$

$$\left[\begin{array}{c}\mathbf{n}\\ \mathbf{k}\end{array}\right]_\mathbf{q}=\frac{[n]_q!}{[k]_q![n-k]_q!}\text{ for }k\in\{0,1,\dots,n\}.$$

The numbers $\left[\begin{array}{c}\mathbf{n}\\ \mathbf{k}\end{array}\right]_q$, $0\leq k\leq n$, are called Gaussian - coefficients.

Let q be an arbitrary complex number, $q\neq 1$, and $\mathcal{D}=\mathcal{D}_q\subseteq\mathbb{C}$ with the properties $x\in\mathcal{D}$ implies $qx\in\mathcal{D}$.

Definition 1. *A function $f:\mathcal{D}_q\to\mathbb{C}$ is said to be q - differentiable, iff $0\in\mathcal{D}_q$, implies that $f'(0)$ exists.*

Definition 2. *A function $f:\mathcal{D}_q\to\mathbb{C}$ is sai to be q - differentiable of order n, iff $0\in\mathcal{D}_q$, implies that $f^{(n)}(0)$ exists.*

For a function $f:\mathcal{D}_q\to\mathbb{C}$ which is q - differentiable its q - derivative $D_q f$ was defined in 1908 by F.H. Jackson [6], in the following way

$$(\mathbf{D_q f})(\mathbf{x})=\frac{f(x)-f(qx)}{(1-q)x},\quad q\in\mathbb{C}\backslash\{1\}. \tag{1}$$

For instance:

$$D_q(x^n)=\frac{x^n-(qx)^n}{(1-q)x}=\frac{x^n(1-q^n)}{x(1-q)}=[n]_q x^{n-1}.$$

and the linear operator $f\to D_q f$ satisfied the relations [2]

$$(D_q fg)(x)=g(x)(D_q f)(x)+f(qx)(D_q g)(x). \tag{2}$$

$$\left(D_q\left(\frac{f}{g}\right)\right)(x)=\frac{g(x)(D_q f)(x)-f(x)(D_q g)(x)}{g(x)g(qx)},\quad g(x)g(qx)\neq 0.$$

In 1846 Heine [4] introduced the so - called q-hypergeometric series

$$(3) \qquad {}_2\Phi_1(\alpha, b; c|_q, a) = \sum_{k=0}^{\infty} \frac{< \alpha; q >_k < b; q >_k}{< 1; q >_k < c; q >_k} a^k,$$

where the following notation is used [1]

$$< a; q >_n = \begin{cases} 1, & n = 0; \\ \prod_{k=0}^{n-1} (1 - q^{a+k}), & n = 1, 2, \ldots \end{cases},$$

$$< a; q >_\infty = \prod_{k=0}^{\infty} (1 - q^{a+k}), \quad 0 < |q| < 1,$$

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad 0 < |q| < 1,$$

$$< a; q^j >_n = \begin{cases} 1, & n = 0; \\ \prod_{k=0}^{n-1} (1 - q^{a+kj}), & n = 1, 2, \ldots \end{cases},$$

$$< a; q^j >_\infty = \prod_{k=0}^{\infty} (1 - q^{a+kj}), \ 0 < |q| < 1,$$

$$(a; q^j)_n = \begin{cases} 1, & n = 0; \\ \prod_{k=0}^{n-1} (1 - aq^{kj}), & n = 1, 2, \ldots \end{cases},$$

$$(a; q^j)_\infty = \prod_{k=0}^{\infty} (1 - aq^{kj}), \ 0 < |q| < 1.$$

Starting with the identity

$$(4) \qquad \frac{1}{(1-a)^\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_n}{k!} a^k = {}_2F_1(\alpha, b; b; a), \quad |a| < 1,$$

let $\alpha = nx$, $x \geq 0$. A. Lupaş [8] considers the linear positive operators

$$(5) \qquad (\mathcal{L}_n f)(x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f\left(\frac{k}{n}\right), \quad x \geq 0$$

with $f : [0, \infty) \to \mathbb{R}$, $|f(x)| \leq e^{Ax}$. Further we use the notation $e_0(t) = 1$, $e_j(t) = t^j$, $j = 1, 2, \ldots$. By imposing condition $\mathcal{L}_n e_1 = e_1$ we find $a = \frac{1}{2}$ and Lupaş operators (4) become

$$(6) \qquad (\mathcal{L}_n f)(x) = (2)^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right), \quad x \geq 0.$$

In order to obtain a q - analogue of (6) let us remark that

$$(7) \qquad {}_2\Phi_1(\alpha, b; b|_q, a) = \sum_{k=0}^{\infty} \frac{< \alpha; q >_k < b; q >_k}{< 1; q >_k < b; q >_k} a^k = \sum_{k=0}^{\infty} \frac{< \alpha; q >_k}{< 1; q >_k} a^k = {}_1\Phi_0(\alpha; -|_q; a)$$

On the other hand, the following proposition is known

Theorem 3 (Heine [5]). *For $|a| < 1$, $0 < |q| < 1$, the identity*

$$(8) \qquad \sum_{k=0}^{\infty} \frac{< \alpha; q >_k}{< 1; q >_k} a^k = \frac{(aq^\alpha; q)_\infty}{(a; q)_\infty}.$$

holds.

Letting $q \to 1$, formula (8) is reduces to formula (4).

Further, we consider the family of operators $(L_n^{<a,q>})(x)$, which depend on the a, q, parameters, defined as follows

$$(9) \qquad (L_n^{<a,q>} f)(x) = \frac{(a;q)_\infty}{(aq^{nx};q)_\infty} \sum_{k=0}^{\infty} \frac{< nx; q >_k}{< 1; q >_k} a^k f\left(\frac{[k]_q}{[n]_q}\right).$$

We calculate

$$(L_n^{<a,q>} e_1)(x) = \frac{(a;q)_\infty}{(aq^{nx};q)_\infty} \sum_{k=1}^{\infty} \frac{< nx; q >_k}{< 1; q >_k} a^k \frac{[k]_q}{[n]_q} =$$

$$= \frac{(a;q)_\infty}{(aq^{nx};q)_\infty [n]_q} \sum_{k=1}^{\infty} \frac{[nx]_q [nx+1]_q \dots [nx+k-1]_q}{[1]_q [2]_q \dots [k]_q} a^k [k]_q =$$

$$= \frac{(a;q)_\infty}{(aq^{nx};q)_\infty [n]_q} \sum_{k=0}^{\infty} \frac{[nx]_q [nx+1]_q \dots [nx+k]_q}{[k]_q!} a^{k+1} =$$

$$= \frac{(a;q)_\infty [nx]_q}{(aq^{nx};q)_\infty [n]_q} a \sum_{k=0}^{\infty} \frac{[nx+1]_q \dots [nx+k]_q}{[k]_q!} a^k =$$

$$= \frac{(a;q)_\infty [nx]_q}{(aq^{nx};q)_\infty [n]_q} a \frac{(aq^{nx+1};q)_\infty}{(a;q)_\infty} = \frac{1-q^{nx}}{1-q^n} a \frac{1}{1-aq^{nx}}.$$

We try to determine $a = \bar{a} = \bar{a}(n)$ such that we have $(L_n^{<a,q>} e_1)(x) = [x]_q$. Because

$$(L_n^{<a,q>} e_1)(x) = \frac{1-q^{nx}}{1-q^n} a \frac{1}{1-aq^{nx}},$$

we conclude with $a = \bar{a} = \dfrac{1}{1+q^{nx}}$. Therefore $(L_n^{<\bar{a},q>} e_1)(x) = \dfrac{[nx]_q}{[n]_q}$.

This suggest us to make the substitution $q \rightsquigarrow \bar{q} = q^{\frac{1}{n}}$. This trick is frequently used in q - Calculus. For this value of q we find $(L_n^{<\bar{a},\bar{q}>} e_1)(x) = [x]_q$, and $\dfrac{[k]_{q^{\frac{1}{n}}}}{[n]_{q^{\frac{1}{n}}}} = \left[\dfrac{k}{n}\right]_q$.

2

In the following we put in evidence some approximation properties of operators

$$(\mathcal{L}_n^{<q>} f)(x) \equiv (L_n^{<\bar{a},\bar{q}>} f)(x)$$

having the images

$$(10) \qquad (\mathcal{L}_n^{<q>} f)(x) = \frac{\left(\frac{1}{1+q^x}, q^{\frac{1}{n}}\right)_\infty}{\left(\frac{q^x}{1+q^x}, q^{\frac{1}{n}}\right)_\infty} \sum_{k=0}^{\infty} \frac{< nx, q^{\frac{1}{n}} >_k}{< 1, q^{\frac{1}{n}} >_k} \frac{1}{(1+q^x)^k} f\left(\frac{[k]_{q^{\frac{1}{n}}}}{[n]_{q^{\frac{1}{n}}}}\right).$$

Since

$$< nx, q^{\frac{1}{n}} >_k = (1-q)^k \prod_{j=0}^{k-1} \left[x + \frac{j}{n}\right]_q, \quad < 1, q^{\frac{1}{n}} >_k = (1-q)^k \prod_{j=0}^{k-1} \left[\frac{1+j}{n}\right]_q,$$

let us denote

$$(11) \qquad \varphi_{n,k}(q;x) = \frac{< nx, q^{\frac{1}{n}} >_k}{< 1, q^{\frac{1}{n}} >_k} = \prod_{j=0}^{k-1} \frac{\left[x + \frac{j}{n}\right]_q}{\left[\frac{1+j}{n}\right]_q}.$$

The first $\varphi_{n,k}(q;x)$, $k=0,1,\ldots$ are

$$\varphi_{n,0}(q;x)=1,\ \varphi_{n,1}(q;x)=[x]_q,\ \varphi_{n,2}(q;x)=[x]_q\frac{\left[x+\frac{1}{n}\right]_q}{\left[\frac{2}{n}\right]_q},\ \ldots$$

Using notation (11) operators $(\mathcal{L}_n^{<q>}f)(x)$ are defined by

$$(\mathcal{L}_n^{<q>}f)(x)=\frac{\left(\frac{1}{1+q^x},q^{\frac{1}{n}}\right)_\infty}{\left(\frac{q^x}{1+q^x},q^{\frac{1}{n}}\right)_\infty}\sum_{k=0}^{\infty}\frac{\varphi_{n,k}(q;x)}{(1+q^x)^k}f\left(\left[\frac{k}{n}\right]_q\right). \tag{12}$$

We continue with the definition given by (1), that is

$$(D_qf)(x)=\frac{f(x)-f(qx)}{(1-q)x},\quad q\in\mathbb{C}\backslash\{1\}.$$

For $e_k(t)=t^k$ it follows

$$(D_qe_k)(x)=[k]_qe_{k-1}(x).$$

Makeing the following notation

$$h_\alpha(a)=\frac{1}{(1-a)\cdot\ldots\cdot(1-aq^{\alpha-1})}.$$

we calculate

$$(D_qh_\alpha)(a)=\frac{1}{(1-q)a}[h(a)-h(qa)]=$$

$$=\frac{1}{(1-q)a}\left[\frac{1}{(1-a)\cdot\ldots\cdot(1-aq^{\alpha-1})}-\frac{1}{(1-aq)\cdot\ldots\cdot(1-aq^{\alpha})}\right]=$$

$$=\frac{1}{(1-q)a}\cdot\frac{1-aq^\alpha-1+a}{(1-a)\cdot\ldots\cdot(1-aq^\alpha)}=\frac{1-q^\alpha}{1-q}\cdot\frac{1}{(1-a)\cdot\ldots\cdot(1-aq^\alpha)},$$

that is

$$(D_qh_\alpha)(a)=[\alpha]_qh_{\alpha+1}(a), \tag{13}$$

to obtain the relation

$$(D_q^2h_\alpha)(a)=[\alpha]_q(D_qh_{\alpha+1})(a)=[\alpha]_q[\alpha+1]_qh_{\alpha+2}(a). \tag{14}$$

$h_\alpha(a)$ may be written as

$$h_\alpha(a)=\sum_{k=0}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}a^k.$$

We calculate

$$(D_qh_\alpha)(a)=\sum_{k=0}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}(D_qe_k)(a)=\sum_{k=0}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_qe_{k-1}(a), \tag{15}$$

and

$$(D_q^2h_\alpha)(a)=\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_q[k-1]_qe_{k-2}(a) \tag{16}$$

From (14) and (16) we have

$$[\alpha]_q[\alpha]_qh_{\alpha+2}(a)=\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_q[k-1]_qa^{k-2}. \tag{17}$$

But

$$[k]_q[k-1]_q=\frac{1}{q}([k]_q^2-[k]_q) \tag{18}$$

Thus relation (17) becomes

$$[\alpha]_q[\alpha+1]_q h_{\alpha+2}(a)=\frac{1}{q}\left[\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_q^2a^{k-2}-\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_qa^{k-2}\right],$$

$$qa^2[\alpha]_q[\alpha+1]_q h_{\alpha+2}(a)=\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_q^2a^{k}-\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_qa^{k}.$$

therefore

$$\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_q^2a^{k}=qa^2[\alpha]_q[\alpha+1]_q h_{\alpha+2}(a)+\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_qa^{k}.$$

On the other hand, from (13) and (15) we have

$$[\alpha]_q h_{\alpha+1}(a)=\frac{1}{a}\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_qa^{k}.$$

In order to demonstrate that

$$\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}[k]_q^2a^{k}=qa^2[\alpha]_q[\alpha+1]_q h_{\alpha+2}(a)+a[\alpha]_q h_{\alpha+1}(a). \tag{19}$$

Starting with the formula the operator considered in relation (9), we calculate it for the e_2 function. The operator becomes

$$(L_n^{<a,q>}e_2)(x)=\frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}\sum_{k=1}^{\infty}\frac{<\alpha;q>_k}{<1;q>_k}a^k\left(\frac{[k]_q}{[n]_q}\right)^2\overset{(19)}{=}$$

$$=\frac{qa^2[\alpha]_q[\alpha+1]_q h_{\alpha+2}(a)+a[\alpha]_q h_{\alpha+1}(a)}{h_\alpha(a)[n]_q^2}=\frac{a[\alpha]_q}{(1-aq^\alpha)[n]_q^2}\left[\frac{aq[\alpha+1]_q}{1-aq^{\alpha+1}}+1\right]=$$

$$=\frac{a[\alpha]_q(aq[\alpha+1]_q+1-aq^{\alpha+1})}{(1-aq^\alpha)(1-aq^{\alpha+1})[n]_q^2}=\frac{a[\alpha]_q(1+aq[\alpha]_q)}{(1-aq^\alpha)(1-aq^{\alpha+1})[n]_q^2}.$$

Replacing in the above relation $a=\bar{a}=\dfrac{1}{1+q^{nx}}$ and $\alpha=nx$ we obtain

$$(L_n^{<\bar{a},q>}e_2)(x)=\frac{\dfrac{1}{1+q^{nx}}[nx]_q\left(1+\dfrac{q}{1+q^{nx}}[nx]_q\right)}{[n]_q^2\left(1-\dfrac{q^{nx}}{1+q^{nx}}\right)\left(1-\dfrac{q^{nx+1}}{1+q^{nx}}\right)}=\frac{[nx]_q(1+q^{nx}+q[nx]_q)}{[n]_q^2(1+q^{nx}-q^{nx+1})}=$$

$$=\frac{(1-q^{nx})\left(1-q+q^{nx}-q^{nx+1}+q-q^{nx+1}\right)}{(1-q^n)^2(1+q^{nx}-q^{nx+1})}=\frac{(1-q^{nx})\left(1+q^{nx}-2q^{nx+1}\right)}{(1-q^n)^2(1+q^{nx}-q^{nx+1})}.$$

Replacing q by $\bar{q}=q^{\frac{1}{n}}$ we obtain successively

$$(\mathcal{L}_n^{<q>}e_2)(x)=\frac{(1-q^x)\left(1+q^x-2q^{x+\frac{1}{n}}\right)}{(1-q)^2(1+q^x-q^{x+\frac{1}{n}})}=[x]_q^2+[x]_q\frac{\left[\frac{1}{n}\right]_q q^x(1+q^x)}{1+q^x-q^{x+\frac{1}{n}}}\overset{n\to\infty}{=}[x]_q^2.$$

In (12) it is supposed that $f:[0,\infty)\to\mathbb{R}$, $q\in(0,1)$, $x\in[0,\infty)$. Let us remark that

$$\lim_{n\to\infty,q<1}\mathcal{L}_n^{<q>}f=\mathcal{L}_nf,$$

where $\mathcal{L}_nf$ is the Lupaş operator. According to Heine's identity the series from the right-hand side of (12) is convergent if

$$|f(x)|\le A(1+q^x)^x,\quad\text{with } A=A(f). \tag{20}$$

Condition (20) guarantees us the convergence of (12) series. Thus, we proved the following proposition

Lemma 4. *Let $e_0(t) = 1$, $e_j(t) = t^j$, $j = 1, 2, \ldots$. If $\mathcal{L}_n^{<q>} f$ is defined by (12), then*

$$(\mathcal{L}_n^{<q>} e_j)(x) = e_j([x]_q) \quad \text{for} \quad j \in \{0, 1\},$$

$$\lim_{n\to\infty} (\mathcal{L}_n^{<q>} e_2)(x) = e_2([x]_q) \quad \text{for all } x \geq 0. \tag{21}$$

Moreover, (21) holds uniformly on any interval $[0, M]$ with $M > 0$.

Theorem 5 (P.P. Korovkin)**.** [7] *If $\lim_{n\to\infty} (\mathcal{L}e_j)(x) = x^j$, $j = 0, 1, 2$ then*

$$\lim_{n\to\infty} (\mathcal{L}f)(x) = f(x),$$

for f continuous on $[0, M]$, $M > 0$.

Theorem 6 (A. Lupaş)**.** [9] *If $\lim_{n\to\infty} (\mathcal{L}e_j)(x) = [\varphi(x)]^j$, $j = 0, 1, 2$ then*

$$\lim_{n\to\infty} (\mathcal{L}f)(x) = f(\varphi(x)),$$

for f continuos on $[0, M]$, $M > 0$.

Using Theorem 6 it follows that the following proposition holds

Theorem 7. *Let $f : [0, \infty) \to \mathbf{R}$, $|f(x)| \leq A(1 + q^x)^x$, $A = A(f) > 0$, $f \in C[0, M]$, with $M > 0$. If $\mathcal{L}_n^{<q>}$ are linear positive operators defined as in (12), then*

$$\lim_{n\to\infty} (\mathcal{L}^{<q>} f)(x) = f([x]_q),$$

uniformly on $[0, M]$.

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The query databases given by relation algebra

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Relational algebra is a theoretical language which can constitute a go-off in understand elemental things to querry databases.

Before to generated a querry of datebases in a relational language must review a phase of analysis, for the determination attributes results, connections between tabels and restriction what must respected.

The operations throught informations are processing can bring togheter in a strategy for a correct execution what can put efficient by relational algebra.

The algebraic relational language contains five operation fundamentally - *selection, projection, product cartesian*, *reunion* and *difference* - which accomplish most operations to find records which interest us. On near these, exist another operations, such as: *union, intersection* and *divizion.* These can be the exprimates by five fundamental operation.

Specify the relational algebras are single operators of restriction, which permit a offcut relation, on horizontal line: SELECTION and on sheer PROJECTION and binary operators of extension: JOIN and DIVISION.

SELECTION sort in table only the records what satisfy a condition named among predicate. Of a selection relation R, condition F, whith notation SF(R), he can be define:

SF R |record t|t R andF(t) true

A ordinary notation is: $\mathbf{R_1 \leftarrow SELECTION(R; <boolean\ \ expresion>}$, but for the sake of sciences will detail:

•R is the relation R(A_1, A_2... A_n) above is applied the selection and were A_i are his attribute.

•R_1 is new relation obtained abaft selections, will have same relational schema with $R{-}R_1(A_1, A_{2...}, A_n)$.

• $< boolean\ \ expresion >$can be writed more analytic thus: $< boolean\ \ expresion > =$(term$_1$) and/or (term$_2$)

and/or (term$_k$), where term$_j$= expresion$_1 \theta$ expresion$_2$, *where* expresion$_1$ or expresion$_2$ are calculating

expression departing from the attribute A$_i$ of relations and θ can be one of the operators for comparation.

To illustrate the way wherewith is given the querry of database, there exist sufficient situations informationale using relational algebra. To demonstrate this we will depart from the following structure of databases:

LOCALITATI {CodPost, Loc, Jud}

CLIENTI {CodCl, DenCl, CodFiscal, Adresa, Telefon}

PRODUSE { CodPr, DenPr, UM, Grupa, ProcTVA}

FACTURI {NrFact, DataFact, CodCl, Obs}

LINIIFACT{ NrFact, Linie, CodPr, Cantitate, PretUnit}

To put toghether all this, we discuss following situation: *Which are the insurance invoices in period: 15-23 april 2003?*

First of all is identified in database table from which is extracted the result. And then established the attribute above applied the predicate of selection. Table in which will operate is **FACTURI**. The predicate of selection using the attribute Datafact:

$R_1 \leftarrow SELECTIE(\mathbf{FACTURI}; Datafact >= 15/04/2003 \quad AND \quad Datafact >= 23/04/2003)$

Is obtained this solution:

Through PROJECTION, a relation can be cut on vertical. If selection extracts in a table certain records, base on the conditions verify by one of this attribute values, projection permits the selection into table only fields desirable.

Formal, there is a relation $R(A_1, A_2, .., A_n)$. Projection relations R about more attributes is a relation which is obtained after running two steps:

a) the elimination among attribute A_i those don't specified;

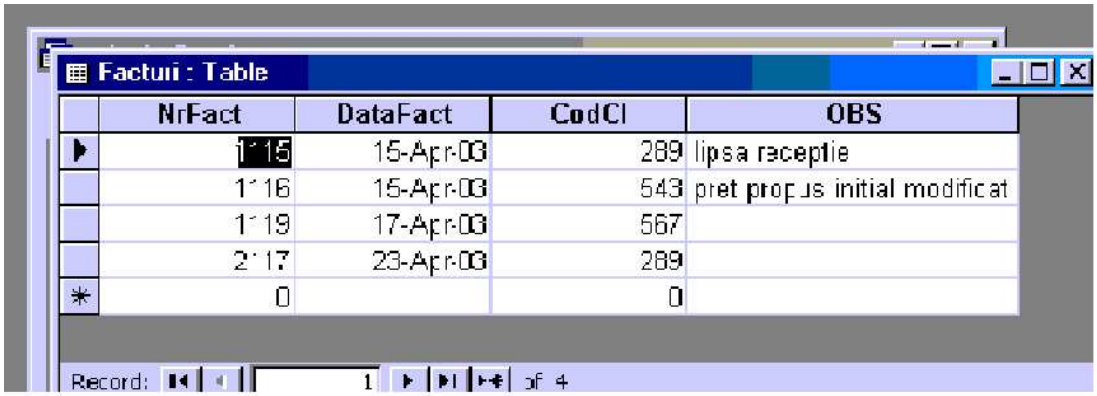

FIGURE 1

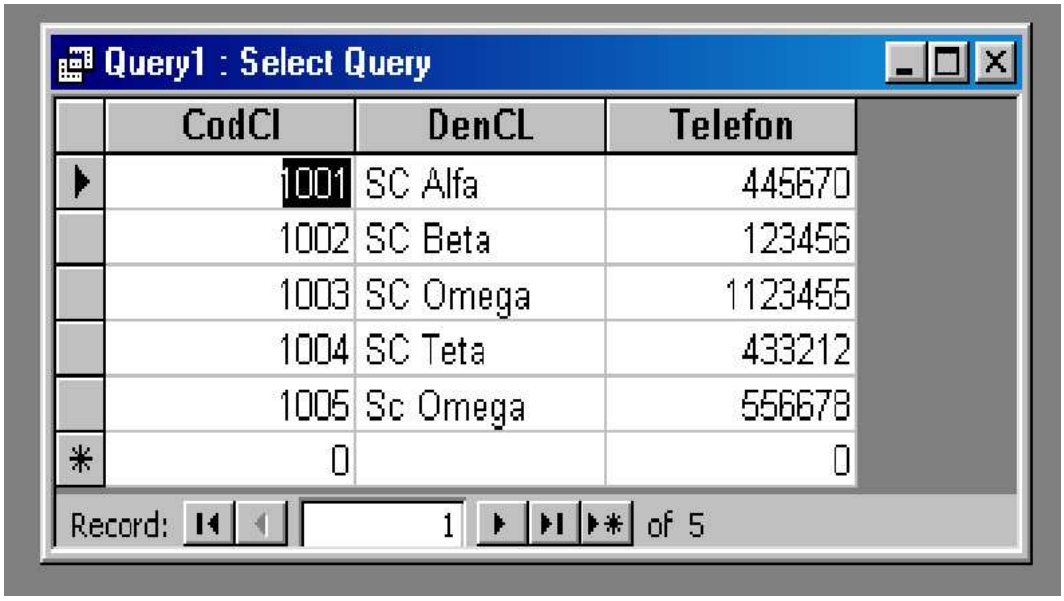

FIGURE 2

b) the suppression dual record (identical recording).

We note: $\mathbf{R}_1 \leftarrow \mathbf{PROJECTION}(\mathbf{R}; \mathbf{A}_j, \mathbf{A}_k, ..., \mathbf{A}_x)$

As opposed to R, relations R_1 is composed only from adequate attribute: $R_1(A_j, A_k, ..., A_x)$. If extraction don't fiind identical records, R_1 have same number of line as the relation R. In contrary case, its number of line is less, depending on the number of duals records.

To illustrate the way how relation algebra determins the query of database choose for exemple, we tries to answer at folowing questions: *What are cod, name, phone number of each custom?* and *What is phone number of custom SC Alfa?*

Table what interest us is CLIENTI, from which it cut out three column: Codcl, Dencl and Telefon.

$\mathbf{R}_1 \leftarrow \mathbf{PROJECTION}(\mathbf{CLIENTI}; \mathbf{CodCl}, \mathbf{DenCl}, \mathbf{Telefon})$

For second situation, whith the aid of selections are cut out from table **CLIENTI** only the proper record for customer solicits.It is obtained a new table and about it is applied a projection, because interest us only the field in which is the call number. Through relational algebra are given the following relations which indicate the enchainmment a selection with a projection.

$\mathbf{R}_1 \leftarrow \mathbf{PROJECTION}(\mathbf{CLIENTI}; \mathbf{DenCl} = \mathbf{SCAlfa}")$

FIGURE 3

$\mathbf{R}_1 \leftarrow$ **PROJECTION**$(\mathbf{R}_1, \mathbf{Telefon})$

The result obtained is a query which cointains the right answer in a new table.

JUNCTION is the derivative operator which replaces the cartesian product because that couldn't be used alone in queries.

If cartesian product is an unconditional fusion between two tables of a database, junction represents the fusion a two tables which have a common property. There are two relations denoted by with: $R_1(A_1, A_2, ..., A_n)$ and $R_2(B_1, B_2, ..., B_p)$. There are A_i and B_j two attribute define on same area, and θ the ensemble comparative operators $\{=, >, <, ?, =, =\}$ what could be applied on two attributs A_i and B_j.

JUNCTION of the relation R_1, throught A_i, whith relation R_2 throught B_j denoteed by: $R_1(A_i \theta B_j)$ R_2 is the relation whom records are obtained concatenating each record of relation R_1 whith records of R_2, if it is checked condition θ established between A_i and B_j. $R_1(A_i \theta B_j)$ $R2 = |\mathbf{t}|\mathbf{t}$ $\mathbf{R}_1$ $\mathbf{R}_2$ $\mathbf{andt}(\mathbf{A}_i)\mathbf{TtB}_j$

Junction presents a different importance for the enquiry databases because permits the recomposition of the original relation. The relational model is banked on the split databases in relation, so that the level of date redudancy and problems for the update tables is minimized. Most many querries, work with date and predicate simultaneously apply to attributes from two or many tables.

Forwards we will try demonstrates the utility junction throught a example. *In what localities it sold the product" XXX" in period 15-23 April 2003?*

One from the classic questions for the verification ways in were undestand junction or not, is : How much line has result-tabela of junction? In case of a BDR the answer is: how much line has main-table. Answered is correct only when is respecte the referential integrity, thus saied, only when all the values foreign key are find in main-table.

Result relation must contains the values of attributes Localitate from table LOCALITĂŢII. The predicate of selection is applied only that in another two tables: PRODUSE, in which Denpr =" XXX" and FACTURI whom records must let us verify the condition: Datafact >¿= 15/04/2003 AND Datafact >= 23/04/2003).

$\mathbf{R}_1 \leftarrow$ **SELECTION** (**PRODUSE**; $DenPr = XXX$")

$\mathbf{R}_2 \leftarrow$ **JUNCTION** (**R1**, **LINIIFACT**; $CodPr = XXX$")

$\mathbf{R}_3 \leftarrow$ **PROJECTION** ($\mathbf{R}_2$; $NrFact$)

$\mathbf{R}_4 \leftarrow$ **SELECTION** (**FACTURI**; $Datafact >= 15/04/2003 AND Datafact >= 23/04/2003$)

$\mathbf{R}_5 \leftarrow$ **JUNCTION** ($\mathbf{R}_3$, $\mathbf{R}_4$; $NrFact$)

$\mathbf{R}_6 \leftarrow$ **JUNCTION** ($\mathbf{R}_5$; $CLIENTI$; $CodCl$)

$\mathbf{R}_7 \leftarrow$ **PROJECTION** ($\mathbf{R}_6$; $CodPost$)

$\mathbf{R}_8 \leftarrow$ **JUNCTION** ($\mathbf{R}_7$, **LOCALITATI**; $CodPost$)

DIVISION is most complex and more difficult to explain among operators discuss in this paper. Codd imagined it as an inverse operator of cartesian product. For they define, started from two relations

R_1(X,Y) and R_2(Y); first relation has two attribute or group of attribute, note X and Y, as the second only the attribute or group of attribute note with Y(define on same area as in relation R_1). A first restriction: R_2(Y), be the numerator of division, is not empty.

Relational **DIDIVISION** $R_1 \div R_2$ has as output a relation define as the ensemble records $\mathbf{R_1}$ **(X)** for which their cartesian product with $\mathbf{R_2(Y)}$ it is a module of $\mathbf{R_1(X,Y)}$.

Result expressions $R_1 \div R_2$ represent the division quotient, be a relation what can be note $\mathbf{R_3(X)}$. In another formulation, $X_i \in R_3$ if and only if $\forall y_i \in Y \in R_2 \rightarrow \exists (x_i, y_i) \in R_1$.

Relational **DIVISION** is dissimilar to useful for the formulation queries in which appears the clause" *either be"* or *" for all"*. In this context, let discuss next example: *What are the customers for which exist one invoice issue every day?*

In another formulation, interest us customers which bought somethings in all days in which they accomplished the sales. Then, the quotient will be a table with an alone attribute Dencl, and the divider will be a relation composed only one attribute Datafact. If it is respected the model require of relational algebra, can note: R_3(DenCl), R_2(DataFact). Knowing the structure of quotient and divider, can determine the structure of dividend-table: R_1 (DenCl, DataFact) that is tabela will contain the names customers and the days in which exist one little an invoice for the respective customer.

Solution can be reproduce in next steps:

- build the divided-relation:

$\mathbf{R} \leftarrow$ **JUNCTION** (**FACTURI**, **CLIENTI**;$CodCl$)

$\mathbf{R_1} \leftarrow$ **PROJECTION** ($\mathbf{R_1}$; $DenCl, DataFact$)

- build the nominator-relation:

$\mathbf{R_2} \leftarrow$ **PROJECTION** (**FACTURI**; $DataFact$)

- the result is: $R_3 \leftarrow R_1 \div R_2$

Scheme and contained of relations involve in this solution are presented below:

Thence, relational algebra consist in operation which react on one or more relations for define an another relation, without modifications for initial relation. Thus as, as much the operands and result are relations, so that come out of an operation become the entrance for an another. This permits expressions in relational algebra, exactly in same way in which is done enchaing mathematical operations, what facilitates diverse methods wherewith interrogate the database.

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Numerical modelling of large scale transport of contaminant solutes in heterogeneous aquifers using a new particles method

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Abstract. For large scale transport problems in groundwater, the methods which track particles to simulate concentrations are successfully used, mainly when the aquifer properties are spatially heterogeneous. These methods are not concerned with numerical diffusion, occurring in finite difference/element schemes. The limitations are only due to large computation time and memory necessary to achieve statistically reliable results and accurate concentration fields. To overcome these computational limitations we use the new "global random walk" algorithm which produces stable and statistically reliable simulations and enables us to investigate the large time behavior of the effective diffusion coefficients, the concentration fluctuations and the thermodynamic equilibrium.

1. Introduction

The solute transport in groundwater is governed by the high space variability of the geological formations. It was shown that at relatively homogeneous small scales the displacement of the solute molecules can be described by Itô stochastic equations. At a larger scale where Darcy law applies, heuristic justifications were also given for the validity of an Itô equation which plays an important role in explaining the " macrodispersion" and the scale effect [Bhattacharya and Gupta, 1979]. Correspondingly, the behavior of the normalized concentration is described by an advection-diffusion equation (the Fokker-Planck equation), i.e. an Eulerian description of the transport process can be introduced. It is worth noticing that the Itô equation, which describes the movements of the molecules at Darcy scale, does not correspond to the usual " Lagrangian" picture in hydrogeological literature. There [see, e.g. Dagan, 1989], the approach based on the Lagrangian coordinate system, borrowed from the theory of turbulent diffusion [Lundgren and Pointin, 1974], is completed with a diffusive movement of the fluid particles. But the trajectories of a diffusion process are not reversible and there is no transformation from Lagrangian to Eulerian coordinates [Lundgren, 1981], thus the fluid particle is not defined. In this respect, the use of fluid particles and " sub-particles" carrying the solute concentration is conceptually wrong.

The heterogeneity of the aquifer properties is described by random space functions, using geostatistical parameters derived from field measurements. The stochastic analysis of transport aims to check the existence of a upscaled effective advection-diffusion equation for mean concentration field and to quantify the concentration fluctuations [Kapoor and Gelhar 1994]. Both the Lagrangian and the Eulerian approaches require further approximations and simplifications and the results are not always in agreement. Some controversial questions and open problems are summarized in the following.

-The time necessary for diffusive behavior to take place corresponds to tens of heterogeneity correlations lengths in 1-st order approximation [Dagan, 1989; Fiori, 1996] while in 2-nd order approach and numerical simulations [Dentz et al., 2003; Schwarze et al., 2001] it was found to be of thousands correlations lengths.

-In Lagrangian approach [Dagan, 1989; Fiori, 1996] the transverse effective diffusion coefficients equal the local coefficients while in Eulerian approach [Gelhar and Axness, 1983] they are larger.

-Although there is a general agreement that after large travel times, when the solute plumes experience the heterogeneity of the aquifer, the ensemble averaged coefficients are relevant for single

realizations of the aquifers, the meaning of this " ergodic hypothesis" [Dagan, 1991] is not clear and no direct verification of this occurrence was provided.

-The predictions for the long time behavior of the concentration coefficient of variation are different in Lagrangian [Dagan and Fiori 1997] and Eulerian [Kapoor and Gelhar 1994; Kapoor and Kitanidis, 1998] theories and little is known about the existence of a thermodynamic equilibrium state of the transport process.

In the present paper, we use a new formulation of the macrodispersion problem, based on Itô equation [Suciu et al., 2002] to get more insight into the meaning of the effective coefficients and ergodicity. Further, we make a direct investigation of the previously enumerated problems through a numerical modeling of the two-dimensional transport. The numerical model benefits by the powerful algorithm called " global random walk" (GRW) [Vamoş et al., 2003]. This new particles method scatters all particles lying at a given grid point simultaneously. Unlike in usual random walk algorithms where the trajectories of the particles are simulated individually and stored, the GRW provides the whole concentration field at each time step and it is not concerned with limitations related to the number of tracked particles.

2. Stochastic description of transport based on Itô equation

The trajectory $X_i(t)$, $i = 1, 2, 3$, of a solute molecule depends on three factors: the local dispersion with constant coefficient D, described as a Wiener process $\mathbf{w}$, the large scale variability of the velocity field, described by a random field $\mathbf{V}$, and the initial position $\mathbf{x}_0$.

For a given realization of the velocity field, the transport is described by the Itô equation

$$X_i(t) = x_{0i} + Ut + \int_0^t u_i(\mathbf{X}(t'))dt' + (2D)^{\frac{1}{2}} \int_0^t dw_i(t'),$$

where $\mathbf{u} = \mathbf{V} - \mathbf{U}$, is a fluctuation around the mean velocity and $\mathbf{U}$ [Suciu et al., 2002].

The variance of the displacements of the molecule in a given realization of the velocity field is given by $\sigma^2_{X_i}(t) = < [X_i(t) - < X_i(t) >_{\mathbf{w},\mathbf{x}_0}]^2 >_{\mathbf{w},\mathbf{x}_0}$, where $< \cdots >_{\mathbf{w},\mathbf{x}_0}$ denotes the average over the realizations of the local dispersion and over the initial positions. From the properties of the Itô integral [Kloeden and Platen, 1995], the effective coefficients $D^{eff}_{ii} = \lim_{t\to\infty} \sigma^2_{X_i}(t)/(2t)$ can be written like

$$D^{eff}_{ii} = D + D^{adv}_{ii} - D^{cm}_{ii} + \mathcal{M}_{ii}, \tag{1}$$

where

$$D^{adv}_{ii} = \lim_{t\to\infty} \int_0^t \langle u_i(\mathbf{X}(t'))u_i(\mathbf{X}(t))\rangle_{\mathbf{w},\mathbf{x}_0}\, dt', \;\; D^{cm}_{ii} = \lim_{t\to\infty} \int_0^t \langle u_i(\mathbf{X}(t'))\rangle_{\mathbf{w},\mathbf{x}_0} \langle u_i(\mathbf{X}(t))\rangle_{\mathbf{w},\mathbf{x}_0}\, dt',$$

$$\mathcal{M}_{ii} = \lim_{t\to\infty} \left\langle X_{0i}u_i(\mathbf{X}(t)) - \langle X_{0i}\rangle_{\mathbf{x}_0} \langle u_i(\mathbf{X}(t))\rangle_{\mathbf{w},\mathbf{x}_0} \right\rangle_{\mathbf{w},\mathbf{x}_0}$$

are contributions due respectively to the fluctuations of the advective velocity, to the fluctuations of the center of mass velocity, and to the cross correlation between X_{0i} and u_i. The last term, $\mathcal{M}_{ii}$, which is non-vanishing for extended initial plumes, is not accounted for in usual Lagrangian approaches.

The average over the realizations of the random velocity field $< D^{eff}_{ii} >_{\mathbf{v}}$ cancels the contribution of $\mathcal{M}_{ii}$ and from (1) we obtain $< D^{eff}_{ii} >_{\mathbf{v}} = D + < D^{adv}_{ii} >_{\mathbf{v}} - < D^{cm}_{ii} >_{\mathbf{v}}$.

The "ergodic hypothesis" [Dagan, 1989],

$$D^{eff}_{ii} = D \; + \; < D^{adv}_{ii} >_{\mathbf{v}}, \tag{2}$$

assumes not only that the effective coefficients in a given realization equal their estimations, but also that $< D^{cm}_{ii} >_{\mathbf{v}} = 0$. There are no theoretical proofs for the occurrence described by (2).

For mean flow U along the longitudinal axis x_1, finite correlation length, λ_y, and small standard deviations, σ_y, of the logarithm of the hydraulic conductivity, y, Dagan [1989] obtained, in first order

approximations of flow and transport,

$$D_{11}^{eff} = D + U\sigma_y^2\lambda_y,\ D_{22}^{eff} = D_{33}^{eff} = D. \tag{3}$$

Gelhar and Axness [1983] found transverse coefficients greater than D, i.e. non-vanishing values for $< D_{ii}^{adv} >_{\mathbf{v}}$.

3. The global random walk algorithm

The one-dimensional GRW algorithm describes the scattering of the $n(i,k)$ particles from (x_i, t_k) by

$$n(j,k) = \delta n(j,j,k) + \delta n(j+v_j-d,j,k) + \delta n(j+v_j+d,j,k) \text{ and } n(i,k+1) = \sum_j \delta n(i,j,k),$$

where v_j are discrete displacements in a given velocity field, d describes the diffusive jumps, and $\delta n(j+v_j \pm d, j, k)$ are random variables with Bernoulli distribution. The diffusion coefficient D is related to the grid steps through $D = r(d\delta x)^2/(2\delta t)$, where $r \leqslant 1$ is the ration of particles undergoing jumps. For two and three-dimensional cases, the same procedure is repeated for all space directions.

The "reduced fluctuations GRW algorithm" is defined by

$$\delta n(j+v_j-d,j,k) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ [n/2]+\theta & \text{if } n \text{ is odd} \end{cases},$$

where $n = n(j,k) - \delta n(j,j,k)$, $[n/2]$ is the integer part of $n/2$ and θ is a variable taking the values 0 and 1 with probability $1/2$.

The GRW algorithm does not yield numerical diffusion and converges as $\mathcal{O}(\delta x^2) + \mathcal{O}(1/\sqrt{N})$, i.e. for large numbers of particles, N, the convergence order is $\mathcal{O}(\delta x^2)$, the same as for finite differences [Vamoş et al., 2003].

4. Temporal behavior of the effective coefficients

Two-dimensional velocity fields with constant mean $U = 1\ m/day$, exponential correlated normal y, with correlation length $\lambda_y = 1\ m$ and variance $\sigma_y^2 = 0.1$, were generated using 640 Fourier modes by the Kraichnan procedure (as in [Schwarze et al., 2001; Dentz et al. 2003]). The local dispersion coefficient was chosen to be $D = 0.01\ m^2/day$, a typical value for transport in groundwater, all the particles were initially located at the origin of the grid and the space and time steps were $\delta x_1 = \delta x_2 = 0.25\ m$ and $\delta t = 1\ day$. Similarly to [Roth and Hammel, 1996], " overshoot errors" were corrected by replacing the velocity with its average over a time step, $V_i = (V(x_i) + V(x_i + \delta t V(x_i)))/2$. In [Suciu et al., 2002] it was shown that for large scale simulations and relatively coarse discretization the overshooting does not affect the values of the effective coefficients and the shape of the plume. It was also checked that the combination of transport parameters $\sigma_y^2 = 0.1$ and $D = 0.01\ m^2/day$ leads to symmetric plumes, as they should be in first order approximations. The computations were conducted for dimensionless times $t/U\lambda_y$ corresponding to 5000 correlation lengths, using the reduced fluctuations GRW algorithm. Because the stochastic description in Section 2, as well as the usual Lagrangian approach, are given for unbounded domains, the grid was chosen to be larger than the maximum extension of the plume.

The convergence of simulations for a given realization of the velocity field requires at least $N = 10^{10}$ particles (Fig. 1). Large ensembles of realizations were computed on a parallel machine Cray T3E. The stochastic convergence of ensemble averages requires $S = 2500$ realizations of the velocity field (Fig. 2). While the convergence of the ensemble averages (more precisely, of the advective coefficients $< D_{ii}^{adv} >_{\mathbf{v}}$) was obtained in the past, for tens of correlation lengths, the convergence of large scale transport simulation in given realizations of the velocity field is a new result, for the first time presented here.

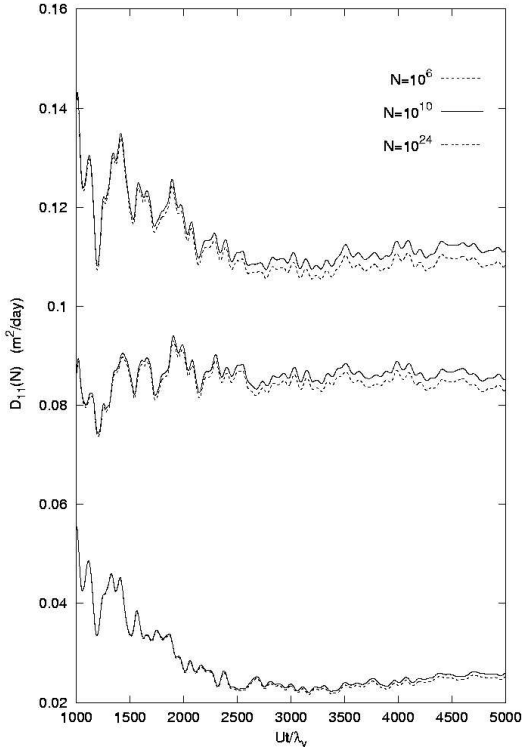

Fig. 1 Convergence as function of N.

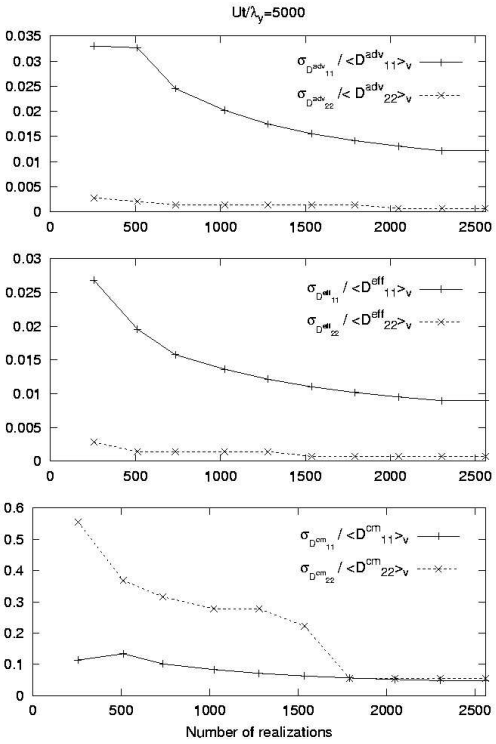

Fig. 2 Convergence of ensemble averages.

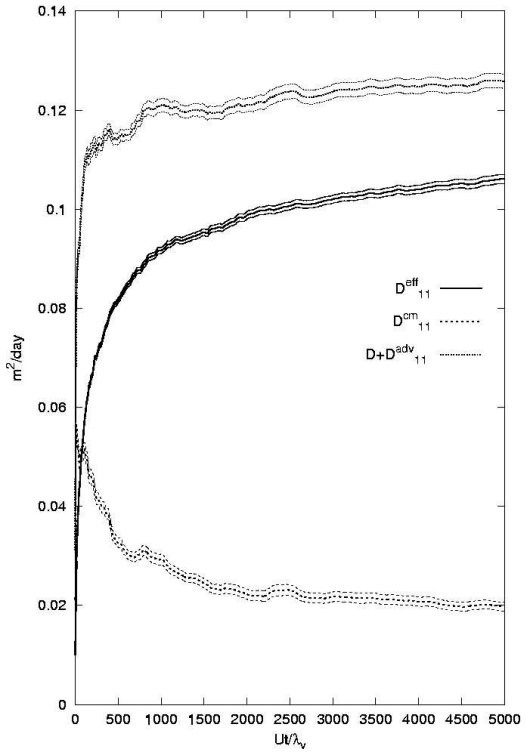

Fig. 3 Longitudianl effective coefficients.

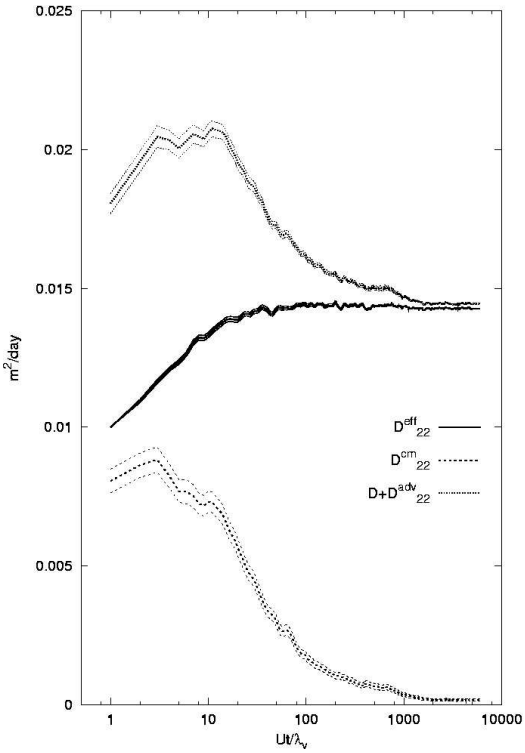

Fig. 4 Transverse effective coefficients.

The longitudinal coefficients are of the same order as predicted by the first order approximation (3) but the ergodic behavior does not occur (the contribution of the center of mass, $< D_{11}^{cm} >_{\mathbf{v}}$ does not vanish), not even for mild variable velocity fields and over thousands correlations lengths (Fig. 3). The ergodicity of the transverse coefficients can be expected to occur at travel times orders of magnitude larger than in first order theory and their asymptotic values are larger than D (Fig. 4).

5. Concentration fluctuations and dilution

The normalized concentration is the probability density of molecules moving on the trajectories $\mathbf{X}(t, \mathbf{x}_0)$ of the transport process described in Section 2 and it is given by the general definition $c(\mathbf{x}, t) = \langle \delta(\mathbf{x} - \mathbf{X}(t, \mathbf{x}_0)) \rangle_{\mathbf{w}, \mathbf{x}_0}$ [van Kampen, 1981]. The average of the concentration over the realizations of the velocity field can be written

$$\langle c(\mathbf{x}, t) \rangle_{\mathbf{v}} = \left\langle \langle \delta(\mathbf{x} - \mathbf{X}(t, \mathbf{x}_0)) \rangle_{\mathbf{w}, \mathbf{x}_0} \right\rangle_{\mathbf{v}} = \int_{\mathcal{V}_0} \langle \delta(\mathbf{x} - \mathbf{X}(t, \mathbf{x}_0)) \rangle_{\mathbf{w}, \mathbf{v}} \, c(\mathbf{x}_0) d\mathbf{x}_0 , \tag{4}$$

where $\mathcal{V}_0$ is the domain occupied by the initial plume and $c(\mathbf{x}_0)$ is the initial concentration field. The average of the square concentration is

$$\left\langle c^2(\mathbf{x}, t) \right\rangle_{\mathbf{v}} = \int_{\mathcal{V}_0} \int_{\mathcal{V}_0} \left\langle \langle \delta(\mathbf{x} - \mathbf{X}(t, \mathbf{x}_{01})) \rangle_{\mathbf{w}} \langle \delta(\mathbf{x} - \mathbf{X}(t, \mathbf{x}_{02})) \rangle_{\mathbf{w}} \right\rangle_{\mathbf{v}} c(\mathbf{x}_{01}) c(\mathbf{x}_{02}) d\mathbf{x}_{01} d\mathbf{x}_{02} \tag{5.a}$$

$$= \int_{\mathcal{V}_0} \int_{\mathcal{V}_0} \langle \delta(\mathbf{x} - \mathbf{X}(t, \mathbf{x}_{01})) \delta(\mathbf{x} - \mathbf{X}(t, \mathbf{x}_{02})) \rangle_{\mathbf{w}, \mathbf{v}} \, c(\mathbf{x}_{01}) c(\mathbf{x}_{02}) d\mathbf{x}_{01} d\mathbf{x}_{02} \tag{5.b}$$

$$= \int_{\mathcal{V}_0} \int_{\mathcal{V}_0} p(\mathbf{x}, t; \mathbf{x}_{01}, 0; \mathbf{x}, t; \mathbf{x}_{02}, 0) c(\mathbf{x}_{01}) c(\mathbf{x}_{02}) d\mathbf{x}_{01} d\mathbf{x}_{02}, \tag{5.c}$$

where (5.a) and (5.b) are identical because the trajectories of the Wiener process starting in two different points $\mathbf{x}_{01}$ and $\mathbf{x}_{02}$ are independent and the function p in (5.c) is a joint density (sometimes called " two particles probability density" [see e.g. Dagan and Fiori, 1997]). From (4) and (5), the variance of the concentration is $\sigma_c^2 = \left\langle c^2(\mathbf{x}, t) \right\rangle_{\mathbf{v}} - (\langle c(\mathbf{x}, t) \rangle_{\mathbf{v}})^2$. In the Lagrangian approach [Dagan and Fiori, 1997] first order results are provided, for which probability density factorizes, $p = p_{\mathbf{v}} \, p_{\mathbf{w}}$, and both $p_{\mathbf{v}}$ and $p_{\mathbf{w}}$ are Gaussian. In the Eulerian approach [Kapoor and Gelhar, 1994, Kapoor and Kitanidis, 1998], σ_c^2 is derived by taking the moments of an advection-diffusion equation supposed to be valid at the Darcy scale and using closure relations to describe the macrodispersion.

The concentration statistics is described by the variance σ_c^2, the concentration coefficient of variation $CV = \sigma_c / \langle c(\mathbf{x}, t) \rangle_{\mathbf{v}}$, computed at the plume center of mass, and the global variance (the integral of σ_c^2 over the solute plume). The dilution of the contaminant solute is described by the " dilution index" E [Kapoor and Kitanidis, 1998],

$$E = \exp\left(- \int c(\mathbf{x}, t) \ln c(\mathbf{x}, t) d\mathbf{x} \right), \tag{6}$$

where the expression under the exponential is the entropy of the process and the integral extends over the entire problem domain. The risk of contamination in a given aquifer should also be quantified by the space mean concentration and variance [Kapoor and Gelhar, 1994]. Additionally, the averages over the realizations of the velocity field account for the incertitude of the space means.

Besides the point source case from previous section, we simulated the two-dimensional transport for extended initial plumes, with and without local dispersion, and the transport in a confined aquifer (for initial plume uniformly distributed over the vertical direction and no flux at upper and lower boundaries, similar to [Kapoor and Kitanidis, 1998]). In all computations the initial plume contained 10^{10} particles and the concentration was computed as the relative number of particles in a square meter. Space statistics were inferred from averages over the transverse dimension of the computation domain and averages over 500 realizations of the velocity were used to estimate the ensemble averages.

The results for unbounded domains from Fig. 5-7 are similar to ones published in the past and underline the crucial role played by the local dispersion (unbounded increase of global variance an CV

in absence of local dispersion). Fig. 7 shows that the ensemble average alone (the points represented by "+") underestimates the incertitude of the spatial distribution of the contaminant.

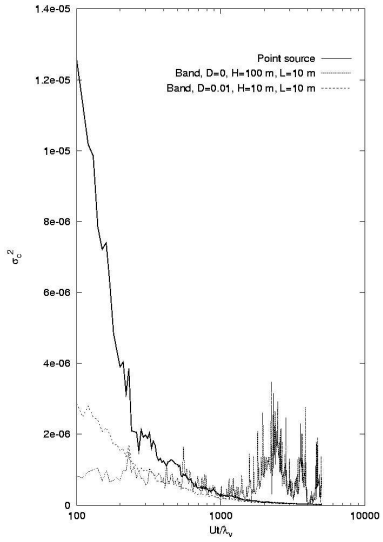

Fig. 5 Concentration variance.

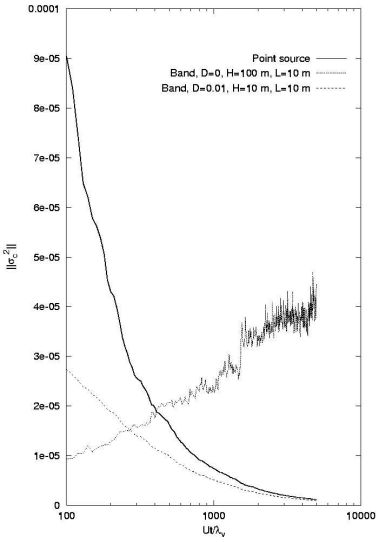

Fig. 6 Global variance.

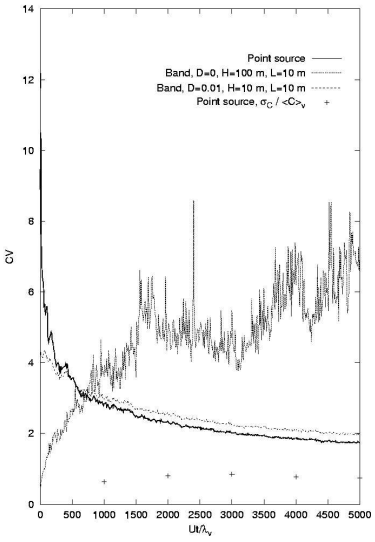

Fig. 7 Concentration coefficient of variation.

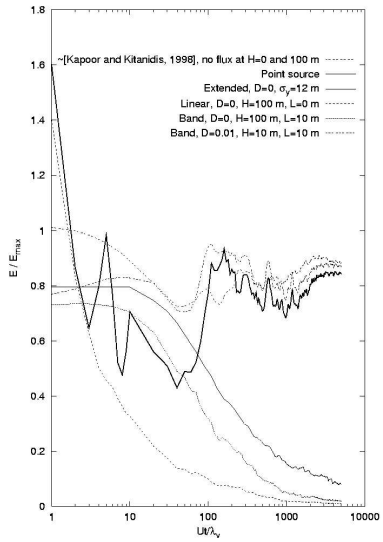

Fig. 8 Normalized dilution index.

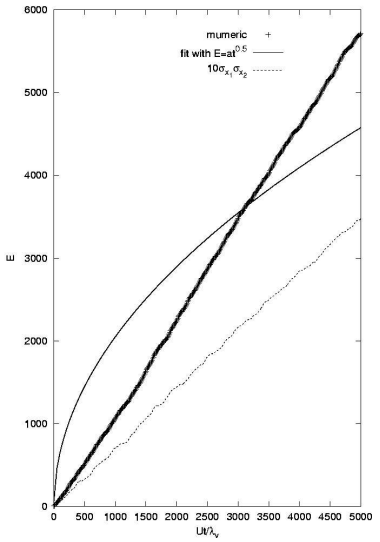

Fig. 9 Unbounded domain.

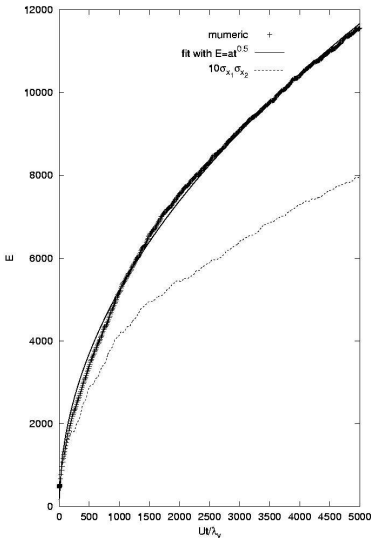

Fig. 10 Confined aquifer.

The dilution index (6), normalized by $E_{\max}$, corresponding to Gaussian distribution, for unbounded domains, and Gaussian distribution on longitudinal direction and uniform on transverse direction, for confined aquifer, is presented in Fig. 8. For both, the behavior is similar, towards $E/E_{\max} = 1$ (occurrence which is called " complete dilution" in [Kapoor and Kitanidis, 1998]), while in the absence of a local dispersion mechanism there is no dilution. Fig. 9 and 10 show that for both boundary problems the dilution index E is proportional to the apparent dimension of the plume $\sigma_{x_1}\sigma_{x_2}$. Since the apparent plume behaves at large times as $\sim t$ for unbounded domains and as $\sim t^{0.5}$ for confined aquifer, the observation of Pannone and Kitanidis [1999] that the results in [Dagan and Fiori, 1997] are in variance with those in [Kapoor and Kitanidis, 1998] is not really surprising. The two results are not

comparable because they are derived from different boundary problems. For both transport processes there is no thermodynamic equilibrium, because the entropy increases unboundedly.

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The mathematical modeling of water flow in open riverbed

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Abstract. Abstract. A macroscopic model of water flow (transfer) along meandering riverbed with discontinous and continous sloping influx of downpour waters has been constructed. The resulted riverbed water discharge function is the solution for the heterogeneous equation of telegraphic type having variable (in time) coefficients. The physical analysis of the solutions showed that, having the minimum empirical data, the proposed model allows us to carry out an estimation of the basic parameters of the movement function of the wave water discharge along the riverbed with different flow routines and distances.

At the last years, the interest for problems connected to environmental protection, rational utilization of natural resources, as well as protection against catastrophic floods has suddenly increased in many countries. This is because the scale of the dangerous consequences of the society's irrational activities and the global, inauspicious changes in the noosphere has been realized. One of these problems, the meeting point of hydrodynamics, wave theory, mass heat transfer, hydraulics, hydrology, meteorology, and other fundamental and applied sciences, is research of floods caused by catastrophic downpour freshets. Solving this hydrological problem is very important to the national economy, as very often the floods cause huge damages and human losses.

A macroscopic model of water flow (transfer) along a meandering riverbed with discontinuous and continuous sloping influx of downpour waters has been constructed. During its construction, there have been used only fundamental concepts inherent to different wave processes, and, first of all, characterizing the kinematics of the process. These are: the (phase or group) wave speed, the extinction coefficient (irreversible losses), the delay, the wave function (solving a telegraphic type equalization), and the radial expansion along the discontinuous and continuous specter of water influx into the riverbed. The minimum of basic notions from hydrology, hydraulics and hydromechanics are applied: stationary (installed) and non-stationary (impulse) flow routines, riverbed section area, water consumption, sloping influx, etc.

At the basis of this model lays the fundamental, and at the same time, the simplest principle, that in proximity applies to different wave processes: if, in point $\overrightarrow{R} = \overrightarrow{R_0}$, the wave function describing the substance movement is ψ_0(t), then in point $\overrightarrow{R} = \overrightarrow{R_1}$ this function is as follows: , where $\psi_1(t) \approx \psi_0(t-\tau_1)e^{-\Theta_1}$, $\tau_1 = \int_{l_0}^{l} \frac{dz}{V(z)}, \Theta_1 = \int_{l_0}^{l} \alpha(l,t)dl$; V$(l,t)$ is the motion speed, $\alpha(l,t)$– the differential coefficient of wave extinction in consequence of losses, l – the length of the water influx axle along which the local wave spreading takes place (linear radial co-ordinate), $\overrightarrow{R}$– radius vector of the observation points. The sign "≈" means that we disregard such "delicate" effects like wave field cross $\vec{r}(\overrightarrow{r} \perp \overrightarrow{l}$),- diffusion, wave profile distortion as a consequence of phase speed dispersion of elementary Fourie-signal, as well as non-linear diffusion (Eilerov whirlwind type of the wave profile).

Utilization of wave-cinematic principle allows to approximately conceive the change function of water consumption in the riverbed influx as a linear superposition of discontinuous and continuous components, conditioned by the local influx of downpour waters into the riverbed system of the reservoir, both in stationary and in un-established routines.

The discontinuous consumption component $Q_d(l,t)$ can be presented by a ultimate radial row composed of N+1 wave functions; it was thorough enough studied earlier (here $Q \approx V_0 \cdot S, V_0$being the medium speed of water movement in the riverbed of the river, S – real section, l – the length of the middle curve of the riverbed, t - time). A more complex problem is calculation of the continuous consumption component $Q_c(l,t)$, more complex than calculation of $Q_d(l,t)$. This is related to the fact that $Q_c(l,t)$ is presented under the form of a curve (Duamel integral or potential lag),

which depends on the density C (l,t) of the continuous (according to l and t) water influx into the riverbed flowing down from the active D area of the reservoir ($D: l \in [a,b]; r \in [-R_2,+R_1]; |D| \equiv mes(D) \approx (b-a)(R_1+R_2)$; R_1 and R_2 are the lengths of the right and left waterside slope in the active riverbed area). To calculate the functions $C(l,t) \sim \frac{\partial}{\partial l}[Q_c(l,r,t)]|_{\tau=0} (l \in [a,b], |t| \leq \infty)$ the aforementioned wave-cinematic principle was used. It allows an approximate description of the flow-down process from the reservoir slopes into the riverbed. This was the purpose of introducing the density (intensity) function of the downpour flows (slope influx models) $d(l,r,t)(\iint_D \int_{-\infty}^{+\infty} d(l,r,t)dl\,dr\,dt = U_0(m^3))$– the general slope influx capacity formed on the D area as a consequence of rainfall. Estimating $d(l,r,t) \approx d_0 f(l) g(r) h(t)(m/s)$ to find $C(l,t) \sim \frac{\partial}{\partial l}[Q_c(l,r,t)]|_{r=0}$ the curve formula $C(l,t) \approx d_0 f(l)\left[\int_0^{R_1} g(r^{'})h(t-r^{'}/u_1)dr^{'} + \int_0^{R_2} g(-r^{'})h(t-r^{'}/u_2)dr^{'}\right]$, was used, where $u_{1,2} \approx v_{1,2}\cos\theta_{1,2}$; $v_{1,2}$- are the speeds of the slope downfall of the downpour waters; $\theta_{1,2}$ are the inclinations of the slopes; d_0 – normalizing coefficient that depends on $U_0, |D|, T, \langle f\rangle, \langle g\rangle, \langle h\rangle$; T – stands for the duration of the rainfall; $\langle f,g,h\rangle$– medium *function value*.

Estimations of $f(l) \sim P_m(l), g(l) \sim P_n(r), h(t) \sim \exp(-|2t/T|^q)$ $(m,n = \overline{0,4}, q = \overline{2,10})$ allowed, with the help of the computer, a qualitative and quantitative research of the formation processes and water discharge impulse structure $Q_c(l,t)$, conditioned by continuous water influx into the riverbed circuit. The resulted riverbed water discharge function is the solution for the heterogeneous equation of the telegraphic type having variable (in time) coefficients. The physical analysis of the solutions showed that, having the minimum empirical data, the proposed model allows us to carry out an estimation of the basic parameters of the movement functions of the wave water discharge along the riverbed with different flow routines and distances. At the same time, some enlargement (according to l) and lengthening (according to t) effects of the impulse $Q_c(l,t)$, have been discovered, these being conditioned by convection processes, i.e. water transportation with different speeds. Extinction and phase-amplitude signal distortions were also studied in different hydraulic routines and riverbed flow parameters.

Notwithstanding the well-known proximity, having the minimal quantity of initial parameters, the proposed wave-cinematic approach allows us to obtain a mathematically correct, achievable and physically content-rich picture of a complex phenomenon of formation, transformation and spreading of wave water discharge in open riverbed. This structural approach can be used for elaboration of computer algorithms allowing to carry out the hydraulic calculations and the prognosis of flood formation on small and large rivers.

Uniform observability of autonomous linear stochastic equations with unbounded coefficients in Hilbert spaces

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Abstract. The problem of the uniform exponential stability and uniform observability of autonomous linear stochastic equations with unbounded coefficients in Hilbert spaces is discussed.

Keywords: stochastic differential equation, uniform exponential stability, uniform observability, Lyapunov equation.

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1. Notation and preliminaries

Let H, V be separable real Hilbert spaces and let $L(H,V)$ be the Banach space of all bounded linear operators from H into V. (If $H=V$ then $L(H,V) \overset{not}{=} L(H)$). We write $\langle .,. \rangle$ for the inner product and $\|.\|$ for norms of elements and operators. We denote by $a \otimes b, a, b \in H$ the bounded linear operator of $L(H)$ defined by $a \otimes b(h) = \langle h, b \rangle a$ for all $h \in H$. The operator $A \in L(H)$ is said to be nonnegative and we write $A \geq 0$, if A is self-adjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in H$. We denote by $L^+(H)$ the subset of $L(H)$ of nonnegative operators.

For $A \in L(H)$, $A \geq 0$ we denote by $A^{1/2}$ the square root of A and by $|A|$ the operator $(A^*A)^{1/2}$.

If $A \in L(H)$ we put $\|A\|_1 = Tr(|A|) \leq \infty$ and we denote by $C_1(H)$ the set $\{A \in L(H) / \|A\|_1 < \infty\}$ (the *operators' trace class*). If $A \in C_1(H)$ we say that A is *nuclear* and it is not difficult to see that A is compact.

The definition of the nuclear operator introduced above is equivalent to that given in [1] and [2]. It is known [1] that $C_1(H)$ is a Banach space endowed with the norm $\|.\|_1$ and for all $A \in L(H)$ and $B \in C_1(H)$ we have $AB, BA \in C_1(H)$.

If $\|A\|_2 = (TrA^*A)^{1/2}$ we can introduce the *Hilbert Schmidt class* of operators, namely $C_2(H) = \{A \in L(H) / \|A\|_2 < \infty\}$ (see [9]). $C_2(H)$ is a Hilbert space with the inner product $\langle A, B \rangle_2 = TrA^*B$ (see [9]).

We denote by $\mathcal{H}_2$ the subspace of $C_2(H)$ of all self-adjoint operators. Since $\mathcal{H}_2$ is closed in $C_2(H)$ with respect to $\|.\|_2$ we deduce that it is a Hilbert space, too. It is known [2] that for all $A \in C_1(H)$ we have $\|A\| \leq \|A\|_2 \leq \|A\|_1$.

Let $(\Omega, F, \mathcal{F}_t, t \in [0,\infty), P)$ be a stochastic basis and $L^2_s(H) = L^2(\Omega, \mathcal{F}_s, P, H)$. If $\xi \in L^2(\Omega, F, P, H)$ we denote by $E(\xi \otimes \xi)$ the bounded linear operator which act on H given by $E(\xi \otimes \xi)(x) = E(\langle x, \xi \rangle \xi)$. The operator $E(\xi \otimes \xi)$, called the *covariance operator* of ξ, is nuclear and $\|E(\xi \otimes \xi)\|_1 = E\|\xi\|^2$ [3].

For each interval $J \subset \mathbf{R}_+ (\mathbf{R}_+ = [0,\infty))$ we denote by $C_s(J, L(H))$ the space of all mappings $G(t) : J \subset \mathbf{R}_+ \to L(H)$ that are strongly continuous. If E is a Banach space we also denote by $C(J,E)$ the space of all mappings $G(t) : J \subset \mathbf{R}_+ \to E$ that are continuous.

2. A representation of the mild solutions of the stochastic differential equations

Assume $A : D(A) \subset H \to H$ is a generator of a C_0-semigroup $S(.)$ [6]. It is known that there exists $n_0 \in \mathbf{N}$ such as for any $n \in \mathbf{N}$, $n \geq n_0$ we have $n \in \rho(A)$. The operators $A_n = n^2 R(n, A) - nI$ $n \geq n_0$ are called the Yosida approximations of A. We consider the equation

$$(1) \qquad \begin{aligned} dy(t) &= Ay(t)dt + \sum_{i=1}^{m} G_i y(t) dw_i(t), \\ y(s) &= \xi \in H, \end{aligned}$$

and the approximating system

$$(2) \qquad \begin{aligned} dy_n(t) &= A_n y_n(t)dt + \sum_{i=1}^{m} G_i y_n(t) dw_i(t), \\ y_n(s) &= \xi \in H, \end{aligned}$$

where $\xi \in L^2_s(H)$, A is the infinitesimal generator of C_0-semigroup $S(.)$, A_n are the Yosida approximations of A, $G_i \in L(H), i = 1, ..., m$ and w_i are real w_i's are independent real Wiener processes relative to $\mathcal{F}_t$.

It is known [7] that (2) has a unique classical solution and (1) has a unique mild solution in $C([s, T]; L^2(\Omega; H))$ that is adapted to $\mathcal{F}_t$; namely the solution of the equation

$$(3) \qquad y(t) = S(t-s)\xi + \sum_{i=1}^{m} \int_s^t S(t-r) G_i y(r) dw_i(r).$$

We denote by $y(t, s; \xi)$ (respectively $y_n(t, s; \xi)$) the mild solution of (1) (respectively the classical solution of (2)).

Lemma 1 ([8]). *There exists a unique mild (resp. classical) solution to (1) (resp. (2)) and $y_n \to y$ in mean square uniformly on any bounded subset of $[s, \infty]$.*

Consider the Lyapunov equation

$$(4) \qquad \frac{dQ(s)}{ds} + A^* Q(s) + Q(s)A + \sum_{i=1}^{m} G_i^* Q(s) G_i + B = 0, s \geq 0$$

and the approximating Lyapunov equations:

$$(5) \qquad \frac{dQ_n(s)}{ds} + A_n^* Q_n(s) + Q_n(s)A_n + \sum_{i=1}^{m} G_i^* Q_n(s) G_i + B = 0, s \geq 0.$$

where $A_n, n \in \mathbf{N}, n \geq n_0$ are the Yosida approximations of A and $B \in L^+(H)$.

According [8], we say that Q is a mild solution on an interval $J \subset \mathbf{R}_+$ of (4), if $Q \in C_s(J, L^+(H))$ and if it satisfies

$$(6) \qquad Q(s)x = S^*(t-s)Q(t)S(t-s)x + \int_s^t S^*(r-s)[\sum_{i=1}^{m} G_i^* Q(r) G_i + B] S(r-s)x dr$$

Lemma 2. [8] *Let $0 < T < \infty$ and $R \in L^+(H)$. Then there exists a unique mild(resp. classical) solution Q (resp. Q_n) of (4) (resp. (5)) on $[0, T]$ such that $Q(T) = R$ (resp. $Q_n(T) = R$). They are given by*

$$Q(s)x = S^*(T-s)RS(T-s)x + \int_s^T S^*(r-s)[\sum_{i=1}^{m} G_i^* Q(r) G_i + B] S(r-s)x dr,$$

$$Q_n(s)x = S_n^*(T-s)RS_n(T-s)x + \int_s^T S_n^*(r-s)[\sum_{i=1}^{m} G_i^* Q_n(r) G_i + B] S_n(r-s)x dr$$

and for each $x \in H$, $Q_n(s)x \to Q(s)x$ uniformly on any bounded subset of $[0,T]$. Moreover, if we denote these solutions by $Q(T,s;R)$ and respectively $Q_n(T,s;R)$ then they are monotone in the sense that $Q(T,s;R_1) \leq Q(T,s;R_2)$ if $R_1 \leq R_2$.

For every $n \in \mathbf{N}$, $n \geq n_0$ we define the mapping $L_n : \mathcal{H}_2 \to \mathcal{H}_2$,

$$L_n(P) = A_n P + P A_n^* + \sum_{i=1}^{m} G_i P G_i^*, P \in \mathcal{H}_2.$$

Obviously $L_n \in L(\mathcal{H}_2)$ and

$$L_n^*(P) = A_n^* P + P A_n + \sum_{i=1}^{m} G_i^* P G_i, P \in \mathcal{H}_2. \tag{7}$$

Let us consider the equation

$$\frac{dP_n(t)}{dt} = L_n P_n(t), \quad P_n(s) = R \in \mathcal{H}_2, t \geq s \geq 0 \tag{8}$$

on $\mathcal{H}_2$. The unique classical solution of (8) is $P_n(t) = \mathcal{S}_n(t-s)(R)$, where $\mathcal{S}_n(t-s) \in L(\mathcal{H}_2)$ is the C_0 semigroup generated by L_n and it is not difficult to see that $\frac{\partial \mathcal{S}_n^*(t-s)}{\partial s} = -L_n^* \mathcal{S}_n^*(t-s)$ for all $t \geq s \geq 0$ [6]. Now it is clear that

$$\frac{\partial}{\partial \sigma} \langle \mathcal{S}_n^*(t-\sigma)R, S \rangle_2 = \langle -L_n^* \mathcal{S}_n^*(t-\sigma)R, S \rangle_2, S, R \in \mathcal{H}_2$$

for all $t \geq \sigma \geq 0$. Let us consider $S = x \otimes x, x \in H$. It is easy to see that $\langle Fx, x \rangle = TrF(x \otimes x)$ for all $F \in L(H)$ and $x \in H$. If $F \in \mathcal{H}_2$ then $\langle F, S \rangle_2 = \langle Fx, x \rangle$. Integrating from s to t, we have

$$\langle \mathcal{S}_n^*(t-s)Rx, x \rangle - \langle Rx, x \rangle = \int_s^t \langle L_n^* \mathcal{S}_n^*(t-\sigma)Rx, x \rangle \, d\sigma, R \in \mathcal{H}_2. \tag{9}$$

Let $Q_n(t,s;R)$ be the unique classical solution of (5) with $B = 0$ such as $Q_n(t) = R, R \geq 0$. Since Q_n also satisfies the integral equation (9) we deduce by using the Gronwall's inequality that for all $R \in \mathcal{H}_2, R \geq 0$ and $t \geq s \geq 0$ we have

$$Q_n(t,s;R) = \mathcal{S}_n^*(t-s)(R) = Q_n(t-s,0;R) \tag{10}$$

A consequence of the results of [10] is the following theorem.

Theorem 3. *Let $C \in L(H,V)$. If $y(t,s;\xi), \xi \in L_s^2(H)$ is the mild solution of (1) and $Q(t,s,R)$ is the unique mild solution of (4) with $B = 0$ and the final value $Q(t) = R \geq 0$ then*

a) $\langle E[y(t,s;\xi) \otimes y(t,s;\xi)]u, u \rangle = TrQ(t,s;u \otimes u)E(\xi \otimes \xi)$ for all $u \in H$

b)

$$E \|C\ y(t,s;\xi)\|^2 = TrQ(t,s;C^*C)E(\xi \otimes \xi).$$

Proof. If $u \in H, \xi \in L_s^2(H)$ and $y_n(t,s;\xi)$ is the classical solution of (2) we deduce from [10](Theorem 7) that

$$\langle E[y_n(t,s;\xi) \otimes y_n(t,s;\xi)]u, u \rangle = TrQ_n(t,s;u \otimes u)E(\xi \otimes \xi) \tag{11}$$

and

$$E \|Cy_n(t,s;\xi)\|^2 = TrQ_n(t,s;C^*C)E(\xi \otimes \xi) \tag{12}$$

for all $t \geq s \geq 0$, where we denote by $Q_n(t,s,R)$ the unique mild solution of (5) with $B = 0$ and the final value $Q_n(t) = R \geq 0$. It is known that the map $Tr : C_1(H) \to \mathbf{C}$ is continuous. From Lemma 2 we obtain

$$TrQ_n(t,s;u \otimes u)E(\xi \otimes \xi) \underset{n \to \infty}{\to} TrQ(t,s;u \otimes u)E(\xi \otimes \xi).$$

As $n \to \infty$ in (11), (12) and using Lemma 1 we obtain a) and b). □

By Theorem 3 and (10) we get

$$E\left\|Cy(t,s;x)\right\|^2 = \langle Q(t,s;C^*C)x,x\rangle = \langle Q(t-s,0;C^*C)x,x\rangle \tag{13}$$

for all $C \in L(H,V), x \in H$ and $t \geq s \geq 0$.

3. The uniform exponential stability and the uniform observability

Let us consider the equation (1) and the observation relation

$$z(t) = Cy(t,s,x), \tag{14}$$

where $C \in L(H,V)$ and $y(t,s,x)$ is the mild solution of (1).

Since $y(.,s;x) \in C([s,T];L^2(\Omega,H))$ for all $x \in H$ it follows that $Cy(.,s;x) \in C([s,T];L^2(\Omega,V))$. We note that $t \to E\left\|Cy(t,s;x)\right\|^2$ is continuous on $[s,T]$.

We will use the notation $\{A;G_i\}$ for the equation (1) and $\{A,C;G_i\}$ for the system (1) şi (14).

Definition 4 ([9]). *We say that $\{A;G_i\}$ is uniformly exponential stable if there exist the constants $M \geq 1$, $\omega > 0$ such that $E\left\|y(t,s;x)\right\|^2 \leq Me^{-\omega(t-s)}\left\|x\right\|^2$ for all $t \geq s \geq 0$ and $x \in H$.*

Definition 5 ([4]). *The system $\{A,C;G_i\}$ is uniformly observable if there exist $\tau > 0$ and $\gamma > 0$ such that $E\int\limits_s^{s+\tau}\left\|Cy(t,s;x)\right\|^2 dt \geq \gamma\left\|x\right\|^2$ for all $s \in \mathbf{R}_+$ and $x \in H$.*

If $Q(t,s,R)$ is the unique mild solution of (4) with $B = 0$ and the final value $Q(t) = R \geq 0$ we use Theorem 3 and Definition 4 and we obtain the following proposition.

Proposition 6. *a) Let I be the identity operator on H. The system (1) is uniformly exponentially stable if and only if there exist the constants $M \geq 1$, $\omega > 0$ such that*

$$Q(t,s;I) \leq Me^{-\omega(t-s)}I \text{ for all } t \geq s \geq 0,$$

b) The system $\{A,C;G_i\}$ is uniformly observable if and only if there exist $\tau > 0$ and $\gamma > 0$ such that

$$\int\limits_s^{s+\tau} Q(t,s;C^*C)dt \geq \gamma I \text{ for all } s \in \mathbf{R}_+.$$

We consider the algebraic Lyapunov equation

$$A^*R + RA + \sum_{i=1}^{m} G_i^*RG_i + C^*C = 0. \tag{15}$$

Theorem 7. *We assume $G_i \in L(H)$ and $C \in L(H,V)$. If $\{A,C;G_i\}$ is uniformly observable then the equation $\{A;G_i\}$ is uniformly exponentially stable if and only if the equation (15) has a unique solution R with the property that there exist the positive constants $\widetilde{m}$, $\widetilde{M}$ such that*

$$\widetilde{m}\left\|x\right\|^2 \leq \langle Rx,x\rangle \leq \widetilde{M}\left\|x\right\|^2 \tag{16}$$

for all $x \in H$.

Proof. " $\Rightarrow$ " In the sequel we use the notation Q (respectively Q_n) introduced by Lemma 2 for the mild (respectively classical) solution of the equation (4) (respectively (5)) with final condition and $B = 0$. Let R be the linear, nonnegative operator, which is given by

$$\langle Rx,x\rangle = E\int\limits_s^{\infty}\left\|Cy(r,s;x)\right\|^2 dr.$$

Since $0 \leq \langle Rx,x\rangle \leq \left\|C\right\|^2\int\limits_s^{\infty} Me^{-\omega(r-s)}\left\|x\right\|^2 dr < \left\|C\right\|^2\frac{M}{\omega}\left\|x\right\|^2$ we deduce that R is well defined. Using Theorem 3, Fubini's theorem and (13) we get

$\langle Rx,x\rangle = \int_s^\infty \langle Q(r,s;C^*C)x,x\rangle\, dr = \int_0^\infty \langle Q(u,0;C^*C)x,x\rangle\, du.$

Consequently, R does not depend on s. If $\tau,\gamma>0$ are the constants introduced by Definition 5 we take $\widetilde{m}=\gamma$, $\widetilde{M}=\|C\|^2\frac{M}{\omega}$ and (16) holds.

Let be $T>0$. If $y_n(t,s;\xi)$ is the classical solution of (2) we consider the linear nonnegative operator $R_T^n(s)$ given by $\langle R_T^n(s)x,x\rangle = E\int_s^T \|Cy_n(r,s;x)\|^2\, dr$. From Theorem 3 we deduce $\langle R_T^n(s)x,x\rangle = \int_s^T \langle Q_n(r-s,0;C^*C)x,x\rangle\, dr$. Since the map $p\to Q_n(p,0;C^*C)x$ is Bochner integrable on $[0,T-s]$ we get $R_T^n(s)x=\int_s^T Q_n(r-s,0;C^*C)xdr$. Differentiating the last equality with respect to s we deduce that $R_T^n(s)$ is the unique classical solution of (5) with $B=C^*C$ such that $R_T^n(T)=0$. As $n\to\infty$ by Lemma 2 we deduce that $R_T^n(s)x \underset{n\to\infty}{\to} R_T(s)x=\int_s^T Q(r,s;C^*C)xdr$ for all $x\in H$ and $R_T(s)x$ is the unique mild solution of (4) with $B=C^*C$ such that $R_T(T)=0$. Since the function $T\to R_T(s)$ is increasing and bounded above on $\mathbf{R}_+$, for every s fixed, it is clear that the sequence $\{R_T(s)\}_T$ converges (as $T\to\infty$) in the strongly operator topology to R which is the solution of the following integral equation

$Rx = S^*(t-s)RS(t-s)x+\int_s^t S^*(r-s)[\sum_{i=1}^m G_i^*RG_i+C^*C]S(r-s)xdr.$

Now it is a simple exercise to prove that R is a solution of the algebraic Lyapunov equation (15). The uniqueness follows from (16) and the uniform exponential stability of $\{A;G_i\}$.

"$\Leftarrow$" Let R be the solution of (15) which satisfies (16).

By using Ito's formula for function $F:\mathbf{R}_+\times H\to\mathbf{R}, F(t,x)=\langle Rx,x\rangle$ and the stochastic process $y(t,s;x), h\in H$ and by taking expectations, we have $\langle Rx,x\rangle = E\langle Ry(t,s;x),y(t,s;x)\rangle + E\int_s^t \|Cy(r,s;x)\|^2\, dr$ for all $x\in H$ and $t\geq s\geq 0$.

Now we obtain $\langle Rx,x\rangle \geq \langle Q(s+\tau,s;R)x,x\rangle+\gamma\|x\|^2$ for all $x\in H$ from Theorem 3 and Definition 5. Replacing x by $y(s,p;x), s\geq p\geq 0, x\in H$, taking expectations and using (16) and Theorem 3 it is not difficult to deduce $(1-\frac{\gamma}{\widetilde{M}})\langle Q(s,p;R)x,x\rangle \geq \langle Q(s+\tau,p;R)x,x\rangle$ for all $s+\tau\geq p\geq 0$ and $x\in H$.

Let $t\geq p\geq 0$. Then there exists $n\in\mathbf{N}$, $r\geq 0$ such that $t-p=n\tau+r, 0\leq r<\tau$ and we have (by induction) $\langle Q(t,p;R)x,x\rangle \leq (1-\frac{\gamma}{\widetilde{M}})^n\langle Q(r+p,p;R)x,x\rangle$. From (16) and since $R\to Q(t,p;R)$ is monotone (according Lemma 2) we get $\widetilde{m}E\|y(t,p;x)\|^2\leq (1-\gamma/\widetilde{M})^n\widetilde{M}E\|y(r+p,p;x)\|^2$ for all $t\geq p\geq 0$ and $x\in H$.

It is known that ([6]) if S is a C_0- semigroup then $\|S(t)\|\leq M_0e^{\omega t}$ for all $t\in\mathbf{R}_+$. So it is not difficult to prove (we use (3)) that there exists the positive constant K such that $E\|y(r+p,p;x)\|^2\leq K\|x\|^2$ and for all $t\geq p\geq 0$ and $x\in H$ we get $E\|y(t,p;x)\|^2\leq [(1-\gamma/\widetilde{M})^{1/\tau}]^{t-p}K(\widetilde{M}/\widetilde{m})(1-\gamma/\widetilde{M})^{-1}]\|x\|^2$. □

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Large Fuzzy logic applications in manuscript character recognition

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The purpose of this paper is improving of manuscript character recognition using a fuzzy logic. Model and some procedures are proposed for processing of scanned data to simplify the problem so that it can be easily described in fuzzy rules.

Technical processes which are conventionally difficult to control are generally characterized by multi-variable control or non-linear and time-variant process characteristics, that can only be described inadequately by mathematical models. Fuzzy applications may be employed either in addition to, or instead of, conventional control in these cases. The fuzzy control block with its defined input/output behavior can therefore be used on the same automation systems in the same way. The character recognition problem presents certain obstacles that make pattern matching on a bit-for-bit basis impractical, because the edge of a character segment can show up in two or more data slices, depending on where the slices overlap. As it is known, one approach to recognition would have a program compare scanned characters to templates on a bit-for-bit basis. Clearly, this procedure could often fail. For instance, the program would expect a 1 in slice 1, bit 31 of a character 0, and neither misaligned nor skewed characters would satisfy the expectation. It is proposed another approach would have the program sum of all the bits in each slice and compare the resulting slice totals to corresponding slice totals from templates.

The definition for the character set to be recognized appears likely in fig.1. Let us consider a darkened pixel to be a bit with a logical value of 1.

FIGURE 1

One of the approaches for recognition would have a program compare scanned characters to templates on a bit-for-bit basis. It is clear, this procedure could often fail. Another approach will be applying fuzzy rules in manuscript characters recognition [2].

For creating and applying fuzzy rules some special characteristics will be used. One of these is a transition that is defined as the difference between a current local maximum (or minimum) and the previous local minimum (or maximum). The data preprocessor takes a data slice, obtains its slice total, and compares the magnitude of the slice total to previous slice totals to determine whether it constitutes a new local maximum or minimum. The proceeding will be the following: the program will be used for calculating the sum of all the bits in each slice. As the result the plot of slice total will be obtained. Figure 2 is a plot of the slice totals for the "m" character. Using program results the transition characteristics will be calculated.

The obtaining results serve the base for "fuzzifying" process [1,2]. It is needed to establish a universe of discourse that defines the range of possible values for fuzzy inputs. Once the universe of discourse is defined, fuzzy sets can be created within it. Quantified transitions will form the input to the character recognition fuzzy engine. The fuzzy rules will look something like this: A very large negative transition, followed by a large negative transition, followed by a large positive transition, followed by a very large positive transition, indicates "j" character:

If (X1 = Very_Large_Negative) & (X2 = Large_Negative) &
(X3 = Large_Positive) & (X4 =Very_Large_Positive) then Char = "j".

It is clear, for the all manuscript characters it is required to write the similar rules. It is important to note that there are very slightly oversized or wandered manuscripts.

Therefore, the range of transition magnitudes for all characters must be obtained by experimental training. Some of these results are shown in Table 1.

Character	X1 (min, max)		X2 (min,max)		X3 (min,max)		X4 (min,max)	
o	14	24	-14	-4	-2	8	..	..
f	14	41	-28	-4	-7	33	-22	-4
a	6	23	-19	-5	-7	22	-10	-6
v	8	15	-10	-4	3	13	..	..

TABLE 1

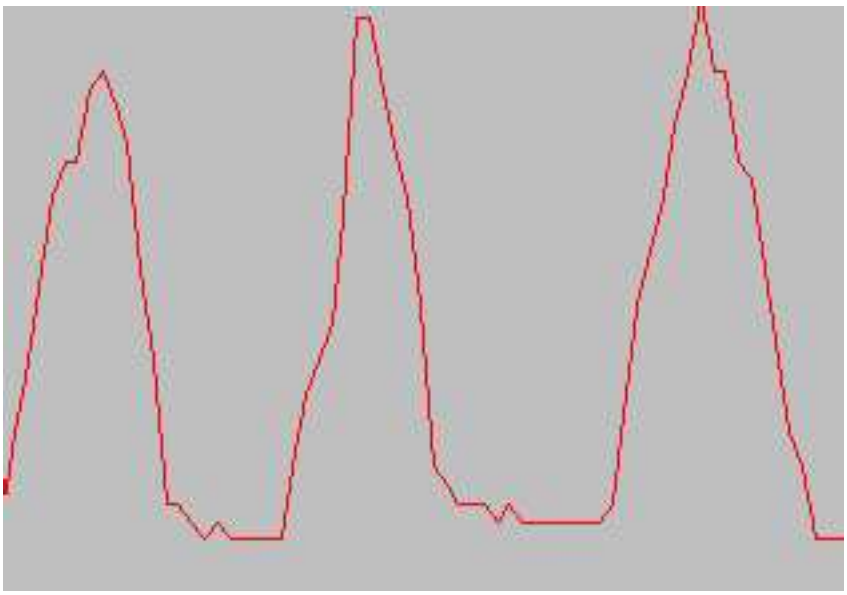

FIGURE 2. Plot of the slice totals of one of "m" character.

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Modelling a producer-consumer system

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Abstract. This paper presents a producer-consumer system with an unlimited buffer modelled by an inhibitor Petri net, and proves some properties of the system.

Keywords: parallel/distributed systems, Petri nets, modelling, verification.

1. Introduction and Preliminaries

A Petri net is a mathematical model used for the specification and the analysis of parallel/distributed systems. An introduction about Petri nets can be found in [Rei85].

One formal analysis method for Petri nets is that of place and transition invariants, which were first introduced in [Lau72]. Place and transition invariants are useful to prove dynamic properties, like reachability, boundedness, home state, liveness and fairness properties.

It is well-known that the behaviour of some distributed systems cannot be adequately modelled by classical Petri nets. Many extensions which increase the computational and expressive power of Petri nets have been thus introduced. One direction has led to various modifications of the firing rule of nets. One of these extensions is that of inhibitor Petri nets.

Let us briefly recall the basic notions and notations concerning Petri nets and inhibitor Petri nets in order to give the reader the necessary prerequisites for the understanding of this paper (for details the reader is referred to [BeF86], [Rei85], [JuT99]). Mainly, we will follow [JuT99].

A *Place/Transition net*, shortly *Petri net*, (finite, with infinite capacities), is a 4-tuple $\Sigma = (S, T, F, W)$, where S and T are two finite non-empty sets (of *places* and *transitions*, resp.), with $S \cap T = \emptyset$, $F \subseteq (S \times T) \cup (T \times S)$ is the *flow relation* and $W : (S \times T) \cup (T \times S) \to \mathbb{N}$ is the *weight function* of Σ verifying $W(x, y) = 0$ iff $(x, y) \notin F$.

A *marking* of a Petri net Σ is a function $M : S \to \mathbb{N}$; it will be sometimes identified with a $|S|$-dimensional vector. The operations and relations on vectors are componentwise defined. $\mathbb{N}^S$ denotes the set of all markings of Σ.

A *marked Petri net* is a pair $\gamma = (\Sigma, M_0)$, where Σ is a Petri net and M_0, called the *initial marking* of γ, is a marking of Σ.

Let Σ be a Petri net, $t \in T$ and $w \in T^*$. The functions $t^-, t^+ : S \to \mathbb{N}$ and $\Delta t, \Delta w : S \to \mathbb{Z}$ are defined by: $t^-(s) = W(s, t)$, $t^+(s) = W(t, s)$, $\Delta t(s) = t^+(s) - t^-(s)$, and

$$\Delta w(s) = \begin{cases} 0, & \text{if } w = \lambda \\ \sum_{i=1}^{n} \Delta t_i(s), & \text{if } w = t_1 t_2 \dots t_n \ (n \geq 1) \end{cases} \text{ , for all } s \in S.$$

The sequential behaviour of a Petri net Σ is given by the so-called *firing rule*, which consists of

- the *enabling rule*: a transition t is *enabled* at a marking M in Σ (or t is *fireable* from M), abbreviated $M[t\rangle_\Sigma$, iff $t^- \leq M$;
- the *computing rule*: if $M[t\rangle_\Sigma$, then t may *occur* yielding a new marking M', abbreviated $M[t\rangle_\Sigma M'$, defined by $M' = M + \Delta t$.

In fact, any transition t of Σ establishes a binary relation on $\mathbb{N}^S$, denoted by $[t\rangle_\Sigma$ and given by: $M[t\rangle_\Sigma M'$ iff $t^- \leq M$ and $M' = M + \Delta t$.

If $t_1, t_2, \dots, t_n$ $(n \geq 1)$ are transitions of Σ, $[t_1 t_2 \dots t_n\rangle_\Sigma$ will denote the classical product of the relations $[t_1\rangle_\Sigma, \dots, [t_n\rangle_\Sigma$. Moreover, the relation $[\lambda\rangle_\Sigma$ is considered, by defining $[\lambda\rangle_\Sigma = \{(M, M) | M \in \mathbb{N}^S\}$.

Let $\gamma = (\Sigma, M_0)$ be a marked Petri net, and $M \in \mathbb{N}^S$. The word $w \in T^*$ is called a *transition sequence* from M in Σ if there exists a marking M' of Σ such that $M[w\rangle_\Sigma M'$. Moreover, the marking

M' is called *reachable* from M in Σ. The set of all reachable markings from M_0 is called *the reachability set* of γ, and it is denoted by $[M_0\rangle_\gamma$.

A place $s \in S$ is *bounded* if there exists $k \in \mathbb{N}$ such that $M(s) \leq k$, for all $M \in [M_0\rangle_\gamma$. The net γ is *bounded* if all its places are bounded.

A transition $t \in T$ is *live* if for any reachable marking $M \in [M_0\rangle_\gamma$, there exists a marking M' reachable from M such that t is fireable from M'. The net γ is *live* if all its transitions are live.

In order to be able to define the notion of the incidence matrix for a Petri net $\Sigma = (S, T, F, W)$, it is necessary to have a total ordering of the sets S and T. Without loss of generality, it will be assumed that, if these sets are of the form

$$S = \{s_1, \ldots, s_m\}, \text{ and } T = \{t_1, \ldots, t_n\},$$

then they are totally ordered by the natural order on the indexes of the elements:

$$S: \ s_1 < \ldots < s_m, \text{ and } T: \ t_1 < \ldots < t_n.$$

The *incidence matrix* of a Petri net Σ is the $m \times n$-dimensional matrix I_Σ defined by

$$I_\Sigma(i,j) = \Delta t_j(s_i), \ \forall\ 1 \leq i \leq m, \ \forall\ 1 \leq j \leq n.$$

The notion of incidence matrix is extended also to marked Petri nets (Σ, M_0) through the unmarked underlying net Σ.

An *S-invariant* (or *place invariant*) of Σ is any m-dimensional vector J of integer numbers which satisfies the equation $J \cdot I_\Sigma = \mathbf{0}$.

The characterization theorem of S-invariants says that, if J is an S-invariant of a marked Petri net $\gamma = (\Sigma, M_0)$, then the relation

$$J \cdot M = J \cdot M_0$$

holds for any $M \in [M_0\rangle_\gamma$. In other words, this theorem says that any S-invariant of γ gives the weights for the places of a subnet of γ in which the tokens are preserved (through these weights).

Inhibitor Petri nets are an extension of Petri nets, which allows them to perform zero tests on locations.

An *inhibitor Petri net* is a pair $\gamma = (\Sigma, I)$, where $\Sigma = (S, T, F, W)$ is a Petri net and $I \subseteq S \times T$, with $F \cap I = \emptyset$, is the set of *inhibitor arcs*.

Let $\gamma = (\Sigma, I)$ be an inhibitor Petri net. The inhibitor arcs $(s,t) \in I$ are also referred to as *zero tests* of γ. More exactly, it is said that the transition t tests for zero the location s. Σ is called the *underlying Petri net* of γ. A *marking* of γ is any marking of its underlying Petri net.

A *marked inhibitor Petri net* is defined similarly as a marked Petri net, by changing "Σ" into "Σ, I".

Pictorially, an inhibitor Petri net will be represented as a classical net and, moreover, the inhibitor arcs I will be drawn as dotted lines, and not as vectors like the normal arcs F.

The behaviour of an inhibitor Petri net $\gamma = (\Sigma, I)$ is given by the *i-firing rule*, consisting of

- the *i-enabling rule*: a transition t is *i-enabled* at a marking M (in γ), abbreviated $M[t\rangle_{\gamma,i}$, iff $M[t\rangle_\Sigma$ and, moreover, $M(s) = 0$ for all $s \in S$ such that $(s,t) \in I$.
- the *i-computing rule*: if $M[t\rangle_{\gamma,i}$, then the marking M' *i-produced* by the occurrence of t at the marking M, abbreviated $M[t\rangle_{\gamma,i}M'$, is defined by $M' = M + \Delta t$ (i.e., like for Petri nets).

The notions of *transition i-sequence* and *i-reachable marking* are defined similarly as for Petri nets. The *set of all i-reachable markings* of a marked inhibitor Petri net γ is denoted by $[M_0\rangle_{\gamma,i}$ (M_0 being the initial marking of γ).

All other notions from Petri nets (i.e. boundedness, liveness, etc.) are defined for inhibitor Petri nets similarly as for Petri nets, by considering the notion of *i-reachability* instead of *reachability* from Petri nets.

The notion of place invariants for inhibitor Petri nets, and results regarding them, are the same as for Petri nets. We will briefly remind this notion.

The *incidence matrix* of an inhibitor Petri net $\gamma = (\Sigma, I)$ is $I_\gamma = I_\Sigma$, where I_Σ is the incidence matrix of the underlying Petri net of γ. The notion of incidence matrix is extended also to marked inhibitor Petri nets (Σ, I, M_0) through the unmarked underlying net (Σ, I).

An *S-invariant* (or *place invariant*) of γ is any S-invariant of its underlying Petri net Σ. The same characterization theorem of S-invariants from Petri nets holds also for inhibitor Petri nets.

The paper is organized as follows. Section 2 presents an example of a producer-consumer system modelled by an inhibitor Petri net, and section 3 presents the verification of the system properties using the place invariant method. Section 4 concludes this paper.

2. Producer-Consumer with unlimited buffer

This section presents an example of using inhibitor Petri nets to model and analyse real systems.

Let us consider a system consisting of a producer and a consumer. The producer produces and sends messages to the consumer, one by one, through an asynchoronous channel (a buffer with unlimited capacity for storing messages). The consumer receives and consumes, one by one, the messages from channel. Moreover, the producer can take a break at any moment, but we impose the restriction that the consumer can enter his inactive state only if the producer is inactive and there is no message pending in the channel.

The same system, but with a limited buffer, was modelled by a Petri net in [Rei85]. Unfortunately, this system with an unlimited buffer cannot be modelled by a Petri net because zero tests of a location with infinite capacity cannot be simulated by Petri nets (a proof of this fact can be found in [JuT99]).

A modelling of this system by an inhibitor Petri net $\gamma = (\Sigma, I, M_0)$ is presented in Figure 1, with the following interpretation of places:

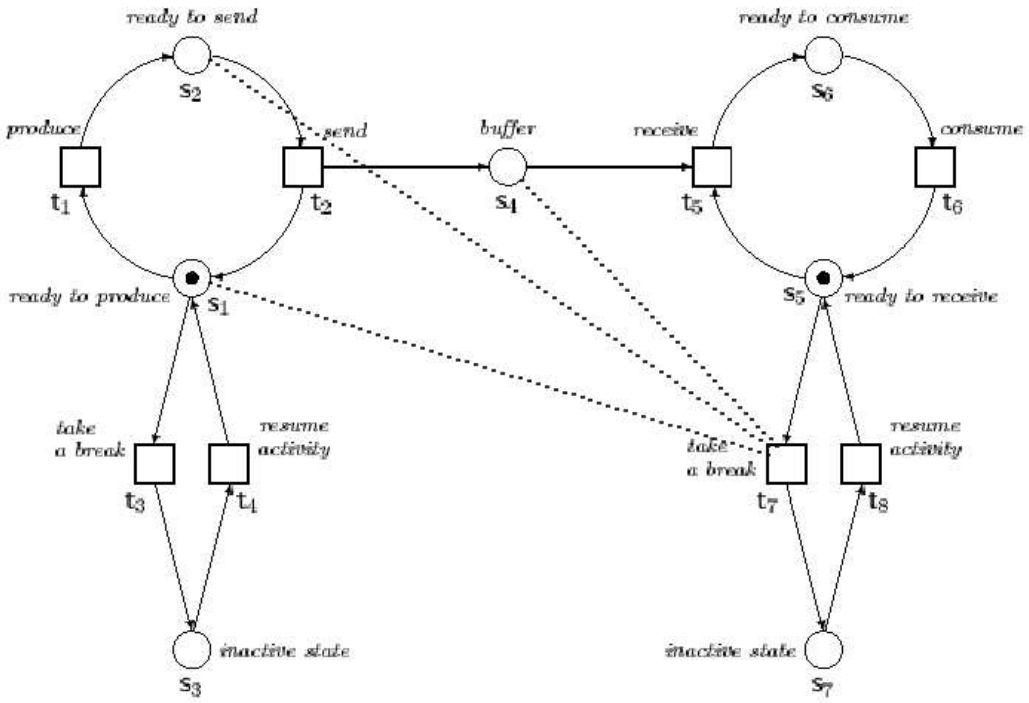

Figure 1. Producer-consumer system with unlimited buffer

- s_1 marked = the producer is ready to produce a message or to take a break;
- s_2 marked = the producer is ready to send the last produced message;
- s_3 marked = the producer is inactive (in a break);
- s_4 = the unlimited buffer for storing messages;
- s_5 marked = the consumer is ready to receive a message or to take a break;
- s_6 marked = the consumer is ready to consume the last received message;

- s_7 marked = the consumer is inactive (in a break).

The interpretations of transition firings are the following:

- t_1 = the producer produces a message;
- t_2 = the producer sends a message;
- t_3 = the producer becomes inactive (takes a break);
- t_4 = the producer resumes his activity;
- t_5 = the consumer receives a message;
- t_6 = the consumer consumes a message;
- t_7 = the consumer becomes inactive (takes a break);
- t_8 = the consumer resumes his activity.

The entering of the consumer in his inactive state, possible only when the producer is inactive and there are no messages in the buffer, is modelled by the three inhibitor arcs of this net, namely the arcs (s_1, t_7) and (s_2, t_7), which tests if producer is inactive, and the arc (s_4, t_7), which tests if there are no messages in the buffer.

We say that the producer-consumer system with an unlimited buffer is *modelled correctly*, if it has the following properties:

- $(\mathbf{P_1})$ At any moment, the producer is in one of the states "ready to produce", "ready to send" or "inactive";
- $(\mathbf{P_2})$ At any moment, the consumer is in one of the states "ready to receive", "ready to consume" or "inactive";
- $(\mathbf{P_3})$ The buffer can contain any number of messages;
- $(\mathbf{P_4})$ The consumer can enter his inactive state only if the producer is in his inactive state and there are no messages in the buffer;
- $(\mathbf{P_5})$ The system is live, i.e. it will never reach a deadlock state.

In the next section we will show how the verification of these properties can be done.

3. Verification of system properties

Using S-invariants, we prove in this section the correctness of our modelling.

Theorem 1. *The inhibitor Petri net from Figure 1 models correctly the producer-consumer system with unlimited buffer.*

Proof. Let $\gamma = (\Sigma, I, M_0)$ be the inhibitor Petri net from Figure 1. It is easy to verify that the vectors

$$J_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

are S-invariants. Moreover, these are the only minimal S-invariants of γ.

Let $M \in [M_0\rangle_{\gamma,i}$ be an arbitrary i-reachable marking of γ. Using the S-invariant J_1 and the characterization theorem of S-invariants, we find that

$$(*) \quad M(s_1) + M(s_2) + M(s_3) = 1,$$

which proves $(\mathbf{P_1})$. Similarly, using J_2 we obtain that

$$(**) \quad M(s_5) + M(s_6) + M(s_7) = 1,$$

which proves $(\mathbf{P_2})$.

In order to prove $(\mathbf{P_3})$, let us notice the following fact. Given any $k \in \mathbb{N}$, by firing the transition sequence $w = (t_1 t_2)^k$ at the marking M_0, a new marking $M \in [M_0\rangle_{\gamma,j}$ will be produced, with $M(s_4) = k$

and $M(s) = M_0(s)$ for all other places of the net. This means that the buffer can contain any number of messages.

In order to prove ($\mathbf{P_4}$), let us notice that, if M is an arbitrary i-reachable marking in which the consumer is inactive (i.e. $M(s_7) = 1$), then M can be reached only by the occurence of transition t_7 of the net γ. It is obviously that transition t_7 of γ can occur only if the producer is inactive and the message channel is empty, because of the inhibitor arcs of γ and the equality $(*)$.

For proving the net γ is live, i.e. it never reaches a deadlock state, we will show that at any i-reachable marking $M \in [M_0\rangle_{\gamma,i}$ there exists at least one transition of γ which is fireable at M. Indeed, from the equality $(*)$ follows that either the transitions t_1 and t_3 are fireable at M, if $M(s_1) = 1$, or the transition t_2 is fireable at M, if $M(s_2) = 1$, or the transition t_4 is fireable at M, if $M(s_3) = 1$. Therefore, the net from Figure 1 is live, which proves ($\mathbf{P_5}$).

This concludes the proof of the system properties. □

Let us remark that from the last argument from above follows also that the producer is live (i.e. the net γ w.r.t. the set of transitions $\{t_1, t_2, t_3, t_4\}$ is live).

Moreover, the consumer (i.e. the net γ w.r.t. the set $\{t_5, t_6, t_7, t_8\}$) is not live, but "almost live", i.e. it never deadlocks excepting the case when the producer is active and the message channel is empty. Indeed, from the equality $(**)$ follows that the only possible cases are the following ones:

i) the transition t_8 is fireable at M, if $M(s_7) = 1$;
ii) the transition t_6 is fireable at M, if $M(s_6) = 1$;
iii) the transition t_5 is fireable at M, if $M(s_5) = 1$ and $M(s_4) > 0$;
iv) the transition t_7 is i-fireable at M, if $M(s_5) = 1$, $M(s_4) = 0$ and $M(s_3) = 1$;
v) the case $M(s_5) = 1$, $M(s_4) = 0$ and $M(s_3) = 0$, i.e. the case in which the producer is active ("ready to produce" or "ready to send") and the message channel is empty, is the only case when the consumer has no directly possible action, but only after an action of the producer (either the producing of a message, or the sending of a message, or the entering of the producer in his inactive state).

4. Conclusion

In this paper we have modelled a sender-receiver system with an unlimited buffer by an inhibitor Petri net, and we have proved the correctness of our modelling by using S-invariants.

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Defining the delays of the asynchronous circuits

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Abstract. The purpose of the paper is that of defining the delays of a circuit as well as the properties of: determinism, order, time invariance, constancy, symmetry and the serial connection.

1. Introduction

Digital electrical engineering is a non-formalized theory and the aim of our concerns is that of trying a semi-formalization. The delay (condition) is the proposed starting point and it represents the real time model of the circuit that computes the identity function $1_{\{0,1\}}$. Logical gates and wires are modeled by a Boolean function that computes instantaneously, in real time, the output depending on the inputs and by zero, one or several delays at the output or at the inputs. The model of an asynchronous circuit consists then in the composition of the models of the logical gates and wires, meaning the serial connection of the delays and the composition of the Boolean functions.

2. Preliminaries

Definition 1. *Let* $\mathbf{B} = \{0,1\}$ *endowed with the discrete topology, with the order* $0 \leq 1$ *and with the usual laws:* $^{-}, \cdot, \cup, \oplus$.

Definition 2. *Let* $x : \mathbf{R} \to \mathbf{B}$ *and* $A \subset \mathbf{R}$. *We define*

$$\bigcap_{\xi \in A} x(\xi) = \begin{cases} 0, \exists \xi \in A, x(\xi) = 0 \\ 1, otherwise \end{cases}, \quad \bigcap_{\xi \in \varnothing} x(\xi) = 1$$

$$\bigcup_{\xi \in A} x(\xi) = \begin{cases} 1, \exists \xi \in A, x(\xi) = 1 \\ 0, otherwise \end{cases}, \quad \bigcup_{\xi \in \varnothing} x(\xi) = 0$$

Definition 3. *The order and the laws of* $\mathbf{B}$ *induce an order and laws in the set of the* $\mathbf{R} \to \mathbf{B}$ *functions, that are noted with the same symbols.*

Definition 4. *Let* $x : \mathbf{R} \to \mathbf{B}$. *The left limit function* $x(t-0)$ *is defined by*

$$\forall t \in \mathbf{R}, \exists \varepsilon > 0, \forall \xi \in (t-\varepsilon, t), x(\xi) = x(t-0)$$

Definition 5. *We suppose that* $x(t-0)$ *exists. Then the functions* $\overline{x(t-0)} \cdot x(t)$, $x(t-0) \cdot \overline{x(t)}$ *are called the left semi-derivatives of* x.

Definition 6. *The characteristic function* $\chi_A : \mathbf{R} \to \mathbf{B}$ *of the set* $A \subset \mathbf{R}$ *is*

$$\chi_A(t) = \begin{cases} 1, t \in A \\ 0, t \notin A \end{cases}$$

Definition 7. *We call signal a function* x *having the property that there exist a unbounded sequence* $0 \leq t_0 < t_1 < t_2 < \ldots$ *so that*

$$x(t) = x(t_0 - 1) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1, t_2)}(t) \oplus \ldots$$

and we note with S *the set of the signals.*

Notation 8. $\tau^d : \mathbf{R} \to \mathbf{R}$ *is the translation* $\tau^d(t) = t - d$, *where* $t, d \in \mathbf{R}$.

Theorem 9. *The constant functions* $0, 1 : \mathbf{R} \to \mathbf{B}$ *are signals. If* $0 \leq m \leq d$ *and* $x, y \in S$, *then the functions* $x \circ \tau^d, \overline{x(t)}, x(t) \cdot y(t), x(t) \cup y(t), x(t) \oplus y(t), \bigcap_{\xi \in [t-d, t-d+m]} x(\xi), \bigcup_{\xi \in [t-d, t-d+m]} x(\xi)$ *are signals too.*

Theorem 10. *For any $x \in S$, the left limit function $x(t-0)$ exists.*

Notation 11. *We note with $P^*(S)$ the set of all non-empty subsets of S.*

3. Stability. Rising and Falling Transmission Delays for Transitions

Definition 12. *Let $u, x \in S$, called input and respectively state (or output). The implication*

$$\forall a \in \mathbf{B}, (\exists t_1, \forall t \geq t_1, u(t) = a) \Longrightarrow (\exists t_2, \forall t \geq t_2, x(t) = a)$$

is called the stability condition (SC). We say that the couple (u, x) satisfies SC. We call also SC the function $Sol_{SC} : S \to P^*(S)$ defined by

$$Sol_{SC}(u) = \{x | (u, x) \ satisfies \ SC\}$$

Definition 13. *We suppose the existence of $a \in \mathbf{B}$ so that $\exists t_1, \forall t \geq t_1, u(t) = a$ and the fact that (u, x) satisfies SC. If u, x are both non-constant, we note*

$$t_1^* = \min\{t_1 | \forall t \geq t_1, u(t) = a\}, \ t_2^* = \min\{t_2 | \forall t \geq t_2, x(t) = a\}$$

The transmission delay for transitions is the number $d \geq 0$ defined by

$$d = \max(0, t_2^* - t_1^*)$$

If $\overline{u(t_1^* - 0)} \cdot u(t_1^*) = \overline{x(t_2^* - 0)} \cdot x(t_2^*) = 1$, then d is called rising and if $u(t_1^* - 0) \cdot \overline{u(t_1^*)} = x(t_2^* - 0) \cdot \overline{x(t_2^*)} = 1$, then d is called falling. If u, respectively x is constant, then t_1^* respectively t_2^* is by definition 0.

4. Delays

Definition 14. *A delay condition (DC) or shortly a delay is a function $i : S \to P^*(S)$ with the property that $\forall u \in S$, $i(u) \subset Sol_{SC}(u)$.*

Remark 15. *The problem of the delays is that of the real time computation of the identity function $1_{\mathbf{B}}$. In practice we often work with systems of equations and inequalities in u, x that model this computation and $i(u)$ represents for all u the set of the solutions of these systems. Definition 14 requests that solutions exist for any u and that the systems be stable.*

Example 16. *The next functions are DC's:*

- $i(u) = \{u\}$ *is usually noted with I. More general, the equation $i(u) = \{u \circ \tau^d\}$ defines a DC noted with $I_d, d \geq 0$.*
- $i(u) = \{x | \exists d \geq 0, x(t) = u(t) \cdot \chi_{[d,\infty)}(t)\}$
- $i(u) = Sol_{SC}(u)$

Theorem 17. *Let $U \subset S$ and the DC's i, j.*

a): If $\forall u, i(u) \wedge U \neq \emptyset$, then the next equation defines a DC

$$(i \wedge U)(u) = i(u) \wedge U$$

b): If i, j satisfy $\forall u, i(u) \wedge j(u) \neq \emptyset$, then $i \wedge j$ is a DC defined by

$$(i \wedge j)(u) = i(u) \wedge j(u)$$

c): Items a), b) are generalized by taking an arbitrary function $\varphi : S \to P^*(S)$ with $\forall u, i(u) \wedge \varphi(u) \neq \emptyset$; $i \wedge \varphi$ is a DC

$$(i \wedge \varphi)(u) = i(u) \wedge \varphi(u)$$

d): i and j define the DC $i \vee j$ in the next manner:

$$(i \vee j)(u) = i(u) \vee j(u)$$

5. Determinism

Definition 18. *The DC i is called deterministic if $\forall u, i(u)$ has a single element and non-deterministic otherwise.*

Remark 19. *By interpreting i as the set of the solutions of a system, its determinism indicates the uniqueness of the solution for all u. On the other hand we shall identify the deterministic DC's with the functions $i : S \to S$. The non-deterministic delays are justified by the fact that in an electrical circuit to one input u there corespond several possible outputs x depending on the variations in ambient temperature, power supply, on the technology etc.*

Example 20. *In 16 I, I_d are deterministic and the other delays are non-deterministic. Let $U \subset S$ and the DC's i, j with i deterministic. If $\forall u, i(u) \wedge U \neq \emptyset$, then $i \wedge U (= i)$ is deterministic and similarly for $i \wedge j$.*

6. The Order

Definition 21. *For the DC's i, j we define*

$$i \subset j \Longleftrightarrow \forall u, i(u) \subset j(u)$$

Remark 22. *The inclusion $\subset$ defines an order in the set of the DC's. Sol_{SC} is the universal element relative to this order, because any i satisfies $i \subset Sol_{SC}$. We interpret the inclusion $i \subset j$ by the fact that the first system contains more restrictive conditions than the second and the model in the first case is more precise than in the second one. In particular, a deterministic DC contains the maximal information and the DC Sol_{SC} contains the minimal information about the modeled circuit.*

Theorem 23. *Any DC j includes a deterministic DC i; if $i \subset j$ and if j is deterministic, then $i = j$.*

7. Time Invariance

Definition 24. *The DC i is called time invariant if*

$$\forall u, \forall x, \forall d \in \mathbf{R}, (u \circ \tau^d \in S \text{ and } x \in i(u)) \Longrightarrow (x \circ \tau^d \in S \text{ and } x \circ \tau^d \in i(u \circ \tau^d))$$

and if the previous property is not satisfied then i is called time variable.

Example 25. *I_d is time invariant, $d \geq 0$. Let the time invariant DC's i, j with $\forall u, i(u) \wedge j(u) \neq \emptyset$; then $i \wedge j$ is time invariant. Let k time invariant; then $i \vee k$ is time invariant. Sol_{SC} is time variable.*

Theorem 26. *If i is a time invariant DC, then the next equivalence holds:*

$$\forall u, \forall x, \forall d \geq 0, x \in i(u) \Longleftrightarrow x \circ \tau^d \in i(u \circ \tau^d)$$

8. Constancy

Definition 27. *A DC i is called constant if $\exists d_r \geq 0, \exists d_f \geq 0$ so that $\forall u, \forall x \in i(u)$ we have*

$$\overline{x(t-0)} \cdot x(t) \leq u(t - d_r)$$
$$x(t-0) \cdot \overline{x(t)} \leq \overline{u(t - d_f)}$$

If the previous property is not satisfied, then i is called non-constant.

Example 28. *I_d is constant, $d \geq 0$. Let $U \subset S$ and the DC's i, j, the first constant. If $i \wedge U$ and $i \wedge j$ are defined, then they are constant. More general, any DC included in a constant DC is constant.*

Theorem 29. *The next functions*

$$x(t) = \bigcap_{\xi \in [t-d, t-d+m]} u(\xi) \quad , \quad x(t) = \bigcup_{\xi \in [t-d, t-d+m]} u(\xi)$$

are deterministic, time invariant, constant DC's, where $0 \leq m \leq d$.

Remark 30. *Constancy means that x is allowed to switch only if u has anticipated this possibility d_r, respectively d_f time units before. Its satisfaction does not imply the uniqueness of d_r, d_f and 29 offers such a counterexample.*

9. Rising-Falling Symmetry

Definition 31. *The DC i is called (rising-falling) symmetrical if*

$$\forall u, i(\overline{u}) = \{\overline{x} | x \in i(u)\}$$

and respectively (rising-falling) asymmetrical otherwise.

Example 32. *$I_d, d \geq 0$ and Sol_{SC} are symmetrical. Let the symmetrical DC's i, j; if $i \wedge j$ is defined, then it is symmetrical. The DC $i \vee j$ is symmetrical too.*

10. The Serial Connection

Definition 33. *For the DC's i, j we note with $k = i \circ j$ the function $k : S \to P^*(S)$ defined by*

$$k(u) = \{y | \exists x, x \in j(u) \text{ and } y \in i(x)\}$$

k is called the serial connection of the DC's i, j.

Theorem 34. *The next statements are true:*

a): *k is a DC.*
b): *If i, j are deterministic, then k is deterministic.*
c): *If i, j are time invariant, then k is time invariant.*
d): *If i, j are symmetrical, then k is symmetrical.*

Remark 35. *The serial connection of the constant delays is not constant, in general. The set of the DC's is a non-commutative semi-group relative to the serial connection and I is the unit.*

Theorem 36. *Let the DC's i, j, k. The next implications are true:*

$$i \subset j \Longrightarrow i \circ k \subset j \circ k$$
$$j \subset k \Longrightarrow i \circ j \subset i \circ k$$

Theorem 37. *Let $U \subset S$ and the DC's i, j, k.*

a): If $\forall u, i(u) \wedge U \neq \emptyset$, then $\forall u, (i \circ j)(u) \wedge U \neq \emptyset$ and

$(i \wedge U) \circ j = (i \circ j) \wedge U$

If $\forall u, j(u) \wedge U \neq \emptyset$, then we have

$i \circ (j \wedge U) \subset i \circ j$

b): If $\forall u, i(u) \wedge j(u) \neq \emptyset$, then $\forall u, (i \circ k)(u) \wedge (j \circ k)(u) \neq \emptyset$ and

$(i \wedge j) \circ k \subset (i \circ k) \wedge (j \circ k)$

If $\forall u, j(u) \wedge k(u) \neq \emptyset$, then $\forall u, (i \circ j)(u) \wedge (i \circ k)(u) \neq \emptyset$ and

$i \circ (j \wedge k) \subset (i \circ j) \wedge (i \circ k)$

c): We have

$(i \vee j) \circ k = (i \circ k) \vee (j \circ k)$

$i \circ (j \vee k) = (i \circ j) \vee (i \circ k)$

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On the inertia of the asynchronous circuits

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Abstract. By making use of the notions and the notation from [12], we present the bounded delays, the absolute inertia and the relative inertia.

1. Bounded Delays

Theorem 1. *The system*

$$\bigcap_{\xi\in[t-d_r,t-d_r+m_r]} u(\xi) \le x(t) \le \bigcup_{\xi\in[t-d_f,t-d_f+m_f]} u(\xi), \tag{1}$$

where $u, x \in S$ *and* $0 \le m_r \le d_r, 0 \le m_f \le d_f$ *defines a DC iff*

$$d_r \ge d_f - m_f, d_f \ge d_r - m_r. \tag{2}$$

Proof. The proof consists in showing that for any u (2) implies the existence of a solution x of (1); any such x satisfies $x \in Sol_{SC}(u)$. If (2) is not fulfilled, it is proved that u exists so that (1) has no solutions. □

Definition 2. *The system (1), when (2) is true, is called the bounded delay condition (BDC).* u, x *are the input, respectively the state (or the output);* m_r, m_f *are the (rising, falling) memories (or thresholds for cancellation) and* d_r, d_f *, respectively* $d_f - m_f, d_r - m_r$ *are the (rising, falling) upper bounds, respectively the (rising, falling) lower bounds of the transmission delay for transitions. We say that the tuple* (u, m_r, d_r, m_f, d_f) *satisfies BDC. We shall also call bounded delay condition the function* $Sol_{BDC}^{m_r,d_r,m_f,d_f} : S \to P^*(S)$ *defined by*

$$Sol_{BDC}^{m_r,d_r,m_f,d_f}(u) = \{x | (u, m_r, d_r, m_f, d_f) \ satisfies \ BDC\}$$

Definition 3. *The inequalities (2) are called the consistency conditions (CC) of BDC.*

Theorem 4. *Let* $0 \le m_r \le d_r, 0 \le m_f \le d_f$ *and* $0 \le m_r^{'} \le d_r^{'}, 0 \le m_f^{'} \le d_f^{'}$ *so that CC is fulfilled for each of them.*

a): We denote $d_r^{"} = \min(d_r, d_r^{'}), d_f^{"} = \min(d_f, d_f^{'}), m_r^{"} = d_r^{"} - \max(d_r - m_r, d_r^{'} - m_r^{'})$, $m_f^{"} = d_f^{"} - \max(d_f - m_f, d_f^{'} - m_f^{'})$. The next statements are equivalent:

a.i) $\forall u, Sol_{BDC}^{m_r,d_r,m_f,d_f}(u) \wedge Sol_{BDC}^{m_r^{'},d_r^{'},m_f^{'},d_f^{'}}(u) \ne \emptyset$,

a.ii) $d_r^{"} \ge d_f^{"} - m_f^{"}, d_f^{"} \ge d_r^{"} - m_r^{"}$

: and if one of them is satisfied, then we have

$$Sol_{BDC}^{m_r,d_r,m_f,d_f} \wedge Sol_{BDC}^{m_r^{'},d_r^{'},m_f^{'},d_f^{'}} = Sol_{BDC}^{m_r^{"},d_r^{"},m_f^{"},d_f^{"}}.$$

b): We use the notations $d_r^{"} = \max(d_r, d_r^{'}), d_f^{"} = \max(d_f, d_f^{'})$, $m_r^{"} = d_r^{"} - \min(d_r - m_r, d_r^{'} - m_r^{'})$, $m_f^{"} = d_f^{"} - \min(d_f - m_f, d_f^{'} - m_f^{'})$. The inequalities $d_r^{"} \ge d_f^{"} - m_f^{"}, d_f^{"} \ge d_r^{"} - m_r^{"}$ are satisfied and

$$Sol_{BDC}^{m_r,d_r,m_f,d_f} \vee Sol_{BDC}^{m_r^{'},d_r^{'},m_f^{'},d_f^{'}} \subset Sol_{BDC}^{m_r^{"},d_r^{"},m_f^{"},d_f^{"}}$$

The previous inclusion becomes equality iff

$$\forall u, Sol_{BDC}^{m_r,d_r,m_f,d_f}(u) \wedge Sol_{BDC}^{m_r^{'},d_r^{'},m_f^{'},d_f^{'}}(u) \ne \emptyset.$$

c): The next statements are equivalent:

c.i) $Sol_{BDC}^{m_r,d_r,m_f,d_f}$ is deterministic;
c.ii) the upper bounds and the lower bounds of the delays coincide

$d_r = d_f - m_f, d_f = d_r - m_r$;

c.iii) the memories are null

$m_r = m_f = 0$;

c.iv) the bounded delay degenerates in a translation

(3) $\exists d \geq 0, Sol_{BDC}^{m_r,d_r,m_f,d_f} = I_d$.

d): The next statements are equivalent:
d.i) $Sol_{BDC}^{m_r,d_r,m_f,d_f} \subset Sol_{BDC}^{m_r^{'},d_r^{'},m_f^{'},d_f^{'}}$;
d.ii) $d_r^{'} - m_r^{'} \leq d_r - m_r \leq d_f \leq d_f^{'},\ d_f^{'} - m_f^{'} \leq d_f - m_f \leq d_r \leq d_r^{'}$.

e): $Sol_{BDC}^{m_r,d_r,m_f,d_f}$ is time invariant.

f): The next statements are equivalent:
f.i) $Sol_{BDC}^{m_r,d_r,m_f,d_f}$ is symmetrical;
f.ii) $d_r = d_f, m_r = m_f$.

g): $Sol_{BDC}^{m_r+m_r^{'},d_r+d_r^{'},m_f+m_f^{'},d_f+d_f^{'}}$ is a BDC and we have

$$Sol_{BDC}^{m_r^{'},d_r^{'},m_f^{'},d_f^{'}} \circ Sol_{BDC}^{m_r,d_r,m_f,d_f} = Sol_{BDC}^{m_r+m_r^{'},d_r+d_r^{'},m_f+m_f^{'},d_f+d_f^{'}}.$$

2. Fixed and Inertial Delays

Definition 5. *Let $u, x \in S$ and let $d \geq 0$. The equation (see 4 (3))*

$$x(t) = u(t-d)$$

is called the fixed delay condition (FDC). The delay defined by this equation is also called pure, ideal or non-inertial. A delay different from FDC is called inertial.

Corollary 6. *FDC is deterministic, time invariant, constant and symmetrical. The serial connection of the FDC's coincides with the composition of the translations*

$$I_d \circ I_{d'} = I_{d'} \circ I_d = I_{d+d'}, d \geq 0, d' \geq 0.$$

Remark 7. *At 5 inertia was defined as the property of the DC's of being not ideal. In particular the non-deterministic DC's, for example the non-trivial BDC's (i.e. the BDC's with memory $m_r + m_f \geqslant 0$), are inertial.*

3. Absolute Inertia

Definition 8. *The property*

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t, t+\delta_r]} x(\xi)$$
$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t, t+\delta_f]} \overline{x(\xi)}$$

true for $\delta_r \geq 0, \delta_f \geq 0$ is called the absolute inertial condition (AIC), or the non-zenoness condition. δ_r, δ_f are called inertial parameters. If it is fulfilled, we say that the tuple (δ_r, δ_f, x) satisfies AIC. We also call AIC the set $Sol_{AIC}^{\delta_r,\delta_f} \subset S$ defined by

$$Sol_{AIC}^{\delta_r,\delta_f} = \{x | (\delta_r, \delta_f, x)\ satisfies\ AIC\}.$$

Remark 9. *AIC means that if x switches from 0 to 1, then it remains 1 at least $\delta_r \geq 0$ time units + the dual property. Remark the trivial situation $\delta_r = \delta_f = 0$.*

Definition 10. *Let i be a DC satisfying $\forall u, i(u) \wedge Sol_{AIC}^{\delta_r,\delta_f} \neq \emptyset$. The DC $i \wedge Sol_{AIC}^{\delta_r,\delta_f}$ is called absolute inertial delay condition (AIDC). $Sol_{BDC}^{m_r,d_r,m_f,d_f} \wedge Sol_{AIC}^{\delta_r,\delta_f}$ is called bounded absolute inertial delay condition (BAIDC).*

Theorem 11. *The numbers $0 \leq m_r \leq d_r, 0 \leq m_f \leq d_f$ with CC true and $\delta_r \geq 0, \delta_f \geq 0$ are given. The next statements are equivalent:*

a): $\forall u, Sol_{BDC}^{m_r,d_r,m_f,d_f}(u) \wedge Sol_{AIC}^{\delta_r,\delta_f} \neq \emptyset$;
b): $\delta_r + \delta_f \leq m_r + m_f$

Corollary 12. $0 \leq m_r \leq d_r, 0 \leq m_f \leq d_f$, $0 \leq m_r^{'} \leq d_r^{'}, 0 \leq m_f^{'} \leq d_f^{'}$ *and* $\delta_r \geq 0, \delta_f \geq 0, \delta_r^{'} \geq 0, \delta_f^{'} \geq 0$ *satisfy* $d_r \geq d_f - m_f, d_f \geq d_r - m_r, d_r^{'} \geq d_f^{'} - m_f^{'}, d_f^{'} \geq d_r^{'} - m_r^{'}, \delta_r + \delta_f \leq m_r + m_f, \delta_r^{'} + \delta_f^{'} \leq m_r^{'} + m_f^{'}$. *In such conditions* $Sol_{BDC}^{m_r,d_r,m_f,d_f} \wedge Sol_{AIC}^{\delta_r,\delta_f}$, $Sol_{BDC}^{m_r^{'},d_r^{'},m_f^{'},d_f^{'}} \wedge Sol_{AIC}^{\delta_r^{'},\delta_f^{'}}, Sol_{BDC}^{m_r+m_r^{'},d_r+d_r^{'},m_f+m_f^{'},d_f+d_f^{'}} \wedge Sol_{AIC}^{\delta_r^{'}\delta_f^{'}}$ *are BAIDC's and the next property of the serial connection holds*

$$(Sol_{BDC}^{m_r^{'},d_r^{'},m_f^{'},d_f^{'}} \wedge Sol_{AIC}^{\delta_r^{'}\delta_f^{'}}) \circ (Sol_{BDC}^{m_r,d_r,m_f,d_f} \wedge Sol_{AIC}^{\delta_r,\delta_f}) \subset$$

$$\subset Sol_{BDC}^{m_r+m_r^{'},d_r+d_r^{'},m_f+m_f^{'},d_f+d_f^{'}} \wedge Sol_{AIC}^{\delta_r^{'}\delta_f^{'}}$$

4. Relative Inertia

Definition 13. $0 \leq \mu_r \leq \delta_r, 0 \leq \mu_f \leq \delta_f$ *and* $u, x \in S$ *are given. The property*

$$\overline{x(t-0)} \cdot x(t) \leq \bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi)$$

$$x(t-0) \cdot \overline{x(t)} \leq \bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{u(\xi)}$$

is called the relative inertial condition (RIC). $\mu_r, \delta_r, \mu_f, \delta_f$ are called inertial parameters. If it is fulfilled, we say that the tuple $(u, \mu_r, \delta_r, \mu_f, \delta_f, x)$ satisfies RIC. We also call RIC the function $Sol_{RIC}^{\mu_r,\delta_r,\mu_f,\delta_f} : S \to P^*(S)$ defined by

$$Sol_{RIC}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) = \{x | (u, \mu_r, \delta_r, \mu_f, \delta_f, x) \text{ satisfies RIC}\}.$$

Theorem 14. *Let $0 \leq \mu_r \leq \delta_r, 0 \leq \mu_f \leq \delta_f, u \in S$ and $x \in Sol_{RIC}^{\mu_r,\delta_r,\mu_f,\delta_f}(u)$ arbitrary. If $\delta_r \geq \delta_f - \mu_f, \delta_f \geq \delta_r - \mu_r$ then $x \in Sol_{AIC}^{\delta_f-\delta_r+\mu_r,\delta_r-\delta_f+\mu_f}$.*

Remark 15. *RIC states that the inertial delays 'model the fact that the practical circuits will not respond (at the output) to two transitions (at the input) which are very close together'* [1], [2]. *Theorem 14 connecting AIC and RIC makes use of the condition $\delta_r \geq \delta_f - \mu_f, \delta_f \geq \delta_r - \mu_r$ that is very similar to CC, but with a different meaning.*

Definition 16. *Let i be a DC with $\forall u, i(u) \wedge Sol_{RIC}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \neq \emptyset$. Then the DC $i \wedge Sol_{RIC}^{\mu_r,\delta_r,\mu_f,\delta_f}$ (see Theorem 4.4 c) in* [12]*) is called relative inertial delay condition (RIDC). In particular $Sol_{BDC}^{m_r,d_r,m_f,d_f} \wedge Sol_{RIC}^{\mu_r,\delta_r,\mu_f,\delta_f}$ is called bounded relative inertial delay condition (BRIDC).*

Theorem 17. *Let be the numbers $0 \leq m_r \leq d_r, 0 \leq m_f \leq d_f$. The conditions are equivalent:*

a) $\forall u, Sol_{BDC}^{m_r,d_r,m_f,d_f}(u) \wedge Sol_{RIC}^{\mu_r,\delta_r,\mu_f,\delta_f}(u) \neq \emptyset$;
b) one of the next conditions is true
b.i): $d_f - m_f \leq \delta_r \leq d_r \leq \delta_r - \mu_r + m_r, d_r - m_r \leq \delta_f \leq d_f \leq \delta_f - \mu_f + m_f$;
b.ii): $d_r - m_r + \mu_r \leq \delta_r \leq d_f - m_f \leq d_r, d_f - m_f + \mu_f \leq \delta_f \leq d_r - m_r \leq d_f$;
b.iii): $d_f - m_f \leq \delta_r \leq d_r - m_r + \mu_r \leq d_r, d_r - m_r \leq \delta_f \leq d_f - m_f + \mu_f \leq d_f$;
b.iv): $\delta_r \leq d_f - m_f \leq \delta_r + m_r - \mu_r \leq d_r, \delta_f \leq d_r - m_r \leq \delta_f + m_f - \mu_f \leq d_f$.

Remark 18. *The equivalent conditions from Theorem 17 are of consistency of BRIDC, they are stronger than CC (of BDC) and weaker than (see the hypothesis $\delta_r \geq \delta_f - \mu_f, \delta_f \geq \delta_r - \mu_r$ from Theorem 14)*

$$d_f - m_f \leq \delta_f - \mu_f \leq \delta_r \leq d_r$$
$$d_r - m_r \leq \delta_r - \mu_r \leq \delta_f \leq d_f$$

Theorem 19. *Let $0 \leq m_r \leq d_r, 0 \leq m_f \leq d_f$ so that CC is fulfilled and $u \in S$ arbitrary. The next statements are equivalent:*

a) $x \in Sol_{BDC}^{m_r,d_r,m_f,d_f}(u) \wedge Sol_{RIC}^{m_r,d_r,m_f,d_f}(u)$;

b)

$$\overline{x(t-0)} \cdot x(t) = \overline{x(t-0)} \cdot \bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi);$$
$$x(t-0) \cdot \overline{x(t)} = x(t-0) \cdot \bigcap_{\xi \in [t-d_f, t-d_f+m_f]} \overline{u(\xi)}.$$

Theorem 20. *Any of the previous equivalent conditions defines a deterministic, time invariant, constant DC.*

Remark 21. *The deterministic situation 19 of BRIDC has as special case I_d, happening when $m_r = m_f = 0, d_r = d_f = d$. On the other hand, the serial connection of the BRIDC's is not a BRIDC. We also mention the possibility of replacing the functions $\bigcap_{\xi \in [t-d_r, t-d_r+m_r]} u(\xi)$, $\bigcup_{\xi \in [t-d_f, t-d_f+m_f]} u(\xi)$ by $\bigcap_{\xi \in [t-d_r, t)} u(\xi)$, $\bigcup_{\xi \in [t-d_f, t)} u(\xi)$ in BDC, the functions $\bigcap_{\xi \in [t, t+\delta_r]} x(\xi)$, $\bigcap_{\xi \in [t, t+\delta_f]} \overline{x(\xi)}$ by $\bigcap_{\xi \in [t, t+\delta_r)} x(\xi)$, $\bigcap_{\xi \in [t, t+\delta_f)} \overline{x(\xi)}$ in AIC, the functions $\bigcap_{\xi \in [t-\delta_r, t-\delta_r+\mu_r]} u(\xi)$ and $\bigcap_{\xi \in [t-\delta_f, t-\delta_f+\mu_f]} \overline{u(\xi)}$ with $\bigcap_{\xi \in [t-\delta_r, t)} u(\xi)$, $\bigcap_{\xi \in [t-\delta_f, t)} \overline{u(\xi)}$ in RIC etc. and some variants of the previous definitions follow. The last six functions are not signals.*

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